

EVIDENCE FOR PARKING CONJECTURES

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ABSTRACT. Let W be an irreducible real reflection group. Armstrong, Reiner, and the author presented a model for parking functions attached to W [2] and made three increasingly strong conjectures about these objects. The author generalized these parking objects and conjectures to the Fuss-Catalan level of generality [26]. Even the weakest of these conjectures would uniformly imply a collection of facts in Coxeter-Catalan theory which are at present understood only in a case-by-case fashion. We prove that when W belongs to any of the infinite families ABCDI, the strongest of these conjectures is generically true.

1. INTRODUCTION

The purpose of this paper is to announce evidence supporting a family of conjectures appearing in [2] and [26] related to generalizations of parking functions from the symmetric group \mathfrak{S}_n to an irreducible real reflection group W . Our most important result is that the strongest of these conjectures holds generically whenever W is not of exceptional type. Let us give some background on and motivation for these conjectures, deferring precise statements of definitions and results to Section 2.

A *(classical) parking function of size n* is a length n sequence (a_1, \dots, a_n) of positive integers whose nondecreasing rearrangement $(b_1 \leq \dots \leq b_n)$ satisfies $b_i \leq i$ for all $1 \leq i \leq n$. The set Park_n of parking functions of size n carries a natural action of the symmetric group \mathfrak{S}_n given by $w.(a_1, \dots, a_n) := (a_{w(1)}, \dots, a_{w(n)})$ for $w \in \mathfrak{S}_n$ and $(a_1, \dots, a_n) \in \text{Park}_n$. Parking functions were introduced by Konheim and Weiss in computer science [18], but have received a great deal of attention in algebraic combinatorics [3, 12, 16].

Parking functions have a natural Fuss generalization. Throughout this paper, we fix a choice $k \in \mathbb{Z}_{>0}$ of Fuss parameter. A *(classical) Fuss parking function of size n* is a length n sequence (a_1, \dots, a_n) of positive integers whose nondecreasing rearrangement $(b_1 \leq \dots \leq b_n)$ satisfies $b_i \leq k(i-1) + 1$ for all $1 \leq i \leq n$. The symmetric group \mathfrak{S}_n acts on the set $\text{Park}_n(k)$ of Fuss parking functions by subscript permutation; when $k = 1$ one recovers $\text{Park}_n(1) = \text{Park}_n$.

In [2], Armstrong, Reiner, and the author presented two generalizations, one algebraic and one combinatorial, of parking functions which are attached to any irreducible real reflection group W . Let h be the Coxeter number of W . The algebraic generalization $\text{Park}_W^{\text{alg}}$ was defined as a certain quotient $\mathbb{C}[V]/(\Theta - \mathbf{x})$ of the coordinate ring $\mathbb{C}[V]$ of the reflection representation V , where $(\Theta - \mathbf{x})$ is an inhomogeneous deformation of an ideal $(\Theta) \subset \mathbb{C}[V]$ arising from a homogeneous system of parameters Θ of degree $h+1$ carrying V^* . The combinatorial generalization $\text{Park}_W^{\text{NC}}$ was defined using a certain W -analog of noncrossing set partitions [5, 24]. The combinatorial model $\text{Park}_W^{\text{NC}}$ is easier to visualize and has connections with W -noncrossing partitions, but the algebraic model $\text{Park}_W^{\text{alg}}$ is easier to understand in a type-uniform fashion.

The combinatorial parking space $\text{Park}_W^{\text{NC}}$ and the algebraic space $\text{Park}_W^{\text{alg}}$ carry actions of not just the reflection group W , but also the product $W \times \mathbb{Z}_h$ of W with an order h cyclic group. Armstrong, Reiner, and the author made a sequence of conjectures (Weak, Intermediate, and Strong) of increasing strength about this action [2]. We refer to these collectively as the Main Conjecture.

Key words and phrases. noncrossing partition, parking function, reflection group.

| type | $k = 1$ | $k \geq 1$ | type | $k = 1$ | $k \geq 1$ |
|----------------------|--------------|--------------|----------------------|----------------|----------------|
| A_1, A_2 | Strong | Strong | A_1, A_2 | Strong | Strong |
| $A_n, n \geq 3$ | Weak | Weak | $A_n, n \geq 3$ | Generic Strong | Generic Strong |
| B_n/C_n | Intermediate | Intermediate | B_n/C_n | Generic Strong | Generic Strong |
| D_n | Intermediate | Intermediate | D_n | Generic Strong | Generic Strong |
| $I_2(m), m \geq 5$ | Strong | Intermediate | $I_2(m), m \geq 5$ | Strong | Generic Strong |
| F_4, H_3, H_4, E_6 | Weak | Open | F_4, H_3, H_4, E_6 | Weak | Open |
| E_7, E_8 | Open | Open | E_7, E_8 | Open | Open |

TABLE 1. The strongest version of the Main Conjecture known in each type, before this paper (left) and after (right). Here $k \geq 1$ is our Fuss parameter.

The Weak Conjecture gives a character formula for the (permutation) action of $W \times \mathbb{Z}_h$ on the combinatorial parking space Park_W^{NC} . The Intermediate and Strong Conjectures assert a strong form of isomorphism $\text{Park}_W^{NC} \cong V^\Theta$ between the combinatorial parking space and a “parking locus” V^Θ attached to the algebraic parking space Park_W^{alg} . The Intermediate Conjecture asserts that this isomorphism holds for one particular choice of the “parameter” Θ , whereas the Strong Conjecture asserts that any choice of Θ would give our isomorphism. Even the Weak Conjecture uniformly implies a collection of uniformly stated facts in Coxeter-Catalan theory which are at present only understood in a case-by-case fashion (see Subsection 2.6 for a statement of these facts).

This setup was extended to the Fuss setting in [26]. The k - W -combinatorial and algebraic parking spaces $\text{Park}_W^{NC}(k)$ and $\text{Park}_W^{alg}(k)$ were defined and specialize as $\text{Park}_W^{NC}(1) = \text{Park}_W^{NC}$ and $\text{Park}_W^{alg}(1) = \text{Park}_W^{alg}$. Both $\text{Park}_W^{NC}(k)$ and $\text{Park}_W^{alg}(k)$ carry actions of the product group $W \times \mathbb{Z}_{kh}$. The definition of $\text{Park}_W^{alg}(k)$ depends on a h.s.o.p. Θ of degree $kh + 1$ carrying V^* and to any such h.s.o.p. Θ we have an associated parking locus $V^\Theta(k)$. The Fuss analog of the Main Conjecture (in its Weak, Intermediate, and Strong incarnations) is presented.

The prior progress on Main Conjecture is presented on the left of Table 1. The assertions for $k = 1$ are proven in [2] and the assertions for $k \geq 1$ are proven in [26]. While these proofs are (of course) case-by-case, uniform evidence for the Main Conjecture in any type W has been discovered which identifies certain “components” of $\text{Park}_W^{NC}(k)$ and $V^\Theta(k)$.

Perhaps the most striking feature of the left of Table 1 is that only the Weak Conjecture is known in type A whereas stronger forms are known for the other infinite families BCDI. That is, the state of knowledge *for the symmetric group* is lacking relative to all other infinite families. The reason for this is that relevant h.s.o.p.’s Θ in type A are harder to write down, making the parking loci $V^\Theta(k)$ attached to symmetric groups harder to access. This is a rare instance where the case of the symmetric group is the most difficult among the real reflection groups! The first main contribution of this paper will remedy this situation.

Theorem 1.1. *The Intermediate Conjecture is true in type A for any Fuss parameter $k \geq 1$.*

The basic idea in the proof of Theorem 1.1 is to pick an arbitrary h.s.o.p. Θ of degree $kh + 1$ carrying V^* such that the corresponding parking locus $V^\Theta(k)$ is reduced. While $V^\Theta(k)$ is hard to understand explicitly, it can be understood indirectly by considering an augmented version of the intersection lattice \mathcal{L} attached to W which includes eigenspaces $E(w, \xi)$ of elements $w \in W$ for eigenvalues $\xi \neq 1$.

Our next contribution is yet another layer of the Main Conjecture which we term the *Generic Strong Conjecture*. The four flavors of the Main Conjecture are related by

$$\text{Strong} \Rightarrow \text{Generic Strong} \Rightarrow \text{Intermediate} \Rightarrow \text{Weak},$$

so that the Generic Strong version sits between the Strong and Intermediate versions.

The Generic Strong Conjecture is easy to conceptualize. The Strong Conjecture states that for *any* h.s.o.p. Θ of degree $kh + 1$ carrying V^* , we have our ‘strong isomorphism’ $\text{Park}_W^{NC}(k) \cong V^\Theta(k)$. The Intermediate Conjecture asserts that *there exists* a choice of Θ so that our isomorphism $\text{Park}_W^{NC}(k) \cong V^\Theta(k)$ holds. The Generic Strong Conjecture states that for a *generic* choice of Θ (understood in an appropriate Zariski sense), we have $\text{Park}_W^{NC}(k) \cong V^\Theta(k)$. While these are three *a priori* distinct conditions, we will prove the following statement uniformly.

Theorem 1.2. *The Intermediate and Generic Strong Conjectures are equivalent for any reflection group W and any Fuss parameter $k \geq 1$.*

The current status of the Main Conjecture is summarized on the right of Table 1. The entries come from combining Theorems 1.1 and 1.2. Our proof of Theorem 1.2 will also show that the only obstacle to proving the Strong Conjecture given the Intermediate Conjecture is the proof of a purely algebraic statement having nothing to do with the combinatorics of noncrossing partitions. Namely, one would only need to show that for any h.s.o.p. Θ of degree $kh + 1$ carrying V^* , the parking locus $V^\Theta(k)$ is reduced. This gives significant evidence for the Strong Conjecture itself.

The proof of Theorem 1.2 is uniform, but not combinatorial. One uses topological and analytic arguments to show that a certain collection of “good” h.s.o.p.’s Θ can be identified with a nonempty Zariski open subset \mathcal{U} of the affine space $\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ which parametrizes all W -equivariant polynomial functions $\Theta : V \rightarrow V$ which are homogenous of degree $kh + 1$. As a Zariski open subspace of an affine complex space, the set \mathcal{U} is path connected in its Euclidean topology. For any path $\gamma : [0, 1] \rightarrow \mathcal{U}$ sending t to Θ_t , one gets a parking locus $V^{\Theta_t}(k)$ for all $0 \leq t \leq 1$. A continuity argument shows that one may “follow group actions along paths” to identify the $W \times \mathbb{Z}_{kh}$ -set structures of these parking loci as t varies.

The remainder of this paper is structured as follows. In **Section 2** we review material on reflection groups and Coxeter-Catalan Theory, recall the main constructions of [2, 26], and state the four flavors of the Main Conjecture. In **Section 3** we present a simple tool (Lemma 3.1) for proving equivariant bijections of G -sets for any group G and give uniform enumerative and algebraic results regarding the actions of $W \times \mathbb{Z}_{kh}$ on $\text{Park}_W^{NC}(k)$ and $V^\Theta(k)$. In **Section 4** we specialize to type A and study the action of $\mathfrak{S}_n \times \mathbb{Z}_{kn}$ on $\text{Park}_{\mathfrak{S}_n}^{NC}(k)$. Using the theory developed in Section 3, this will allow us to prove Theorem 1.1. We return to general type W in **Section 5** with a proof of Theorem 1.2. In particular, all of the arguments appearing in this paper are type A or uniform, and the only type A arguments appear in Section 4.

2. BACKGROUND

2.1. Notation for group actions. Let G be a finite group and let \mathcal{S} and \mathcal{T} be finite G -sets. We write $\mathcal{S} \cong_G \mathcal{T}$ to mean that there is a G -equivariant bijection $\varphi : \mathcal{S} \rightarrow \mathcal{T}$. If V and W are finite-dimensional $\mathbb{C}[G]$ -modules, we write $V \cong_{\mathbb{C}[G]} W$ to mean that there is a G -equivariant linear isomorphism $\varphi : V \rightarrow W$. If \mathcal{S} and \mathcal{T} are finite G -sets, we have that $\mathcal{S} \cong_G \mathcal{T}$ implies $\mathbb{C}[\mathcal{S}] \cong_{\mathbb{C}[G]} \mathbb{C}[\mathcal{T}]$, but the converse does not hold in general.

2.2. Reflection groups. Let W be a reflection group acting on its reflection representation V . In this paper, all reflection groups will be real and irreducible. It will be convenient to replace V with its complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ and regard V as a complex vector space. We let $n := \dim(V)$ be the *rank* of W .

Let $\Phi \subset V$ denote a root system associated to V and let $\Phi^+ \subset \Phi$ be a choice of positive system within Φ . Let $\Pi \subseteq \Phi^+$ be the corresponding choice of simple system. For any $\alpha \in \Phi$, let $H_\alpha \subset V$ denote the orthogonal hyperplane $H_\alpha := \{v \in V : \langle v, \alpha \rangle = 0\}$. The hyperplane arrangement $\text{Cox}(W) := \{H_\alpha : \alpha \in \Phi^+\}$ is called the *Coxeter arrangement* of W .

For any $\alpha \in \Phi^+$, let $t_\alpha \in W$ denote the orthogonal reflection through the hyperplane H_α . The set $S := \{t_\alpha : \alpha \in \Pi\}$ of *simple reflections* generates W and turns the pair (W, S) into a Coxeter system. Let $T = \{t_\alpha : \alpha \in \Phi^+\}$ denote the set of *all reflections* in W , simple or otherwise.

If $S = \{s_1, s_2, \dots, s_n\}$, a *Coxeter element* in W is a W -conjugate of the product $s_1 s_2 \cdots s_n$ (where the simple reflections are taken in some order). It can be shown that any two Coxeter elements in W are conjugate. We fix a choice of Coxeter element $c \in W$. We let h denote the order of the group element $c \in W$; the number h is the *Coxeter number* of W and is independent of our choice of c .

Example 2.1. In type A_{n-1} , we may identify W with the symmetric group \mathfrak{S}_n . The reflection representation V is the $(n-1)$ -dimensional quotient $V = \mathbb{C}^n / \langle (1, 1, \dots, 1) \rangle$ of the defining action of \mathfrak{S}_n on \mathbb{C}^n by the copy of the trivial representation given by constant vectors in \mathbb{C}^n .

If e_i denotes the image in V of the i^{th} coordinate vector in \mathbb{C}^n , the root system Φ is given by $\Phi = \{e_i - e_j : 1 \leq i \neq j \leq n\} \subset V$. The standard choice of positive system $\Phi^+ \subset \Phi$ is $\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n\}$. The corresponding simple system is $\Pi = \{e_i - e_{i+1} : 1 \leq i \leq n-1\}$. The Coxeter arrangement $\text{Cox}(\mathfrak{S}_n)$ is the image in V of the standard braid arrangement $\{x_i - x_j = 0 : 1 \leq i < j \leq n\}$ in \mathbb{C}^n , where x_i is the i^{th} standard coordinate function.

The reflection $t_{\alpha_{i,j}} \in \mathfrak{S}_n$ corresponding to a given positive root $\alpha_{i,j} = e_i - e_j$ is the transposition $(i, j) \in \mathfrak{S}_n$. We get that $S = \{(i, i+1) : 1 \leq i \leq n-1\}$ and $T = \{(i, j) : 1 \leq i < j \leq n\}$. The usual choice of Coxeter element is $c = (1, 2)(3, 4) \cdots (n-1, n) = (1, 2, \dots, n) \in \mathfrak{S}_n$. The Coxeter number is $h = n$.

2.3. The algebraic parking space and parking loci. Let $k \in \mathbb{Z}_{>0}$ be a fixed choice of Fuss parameter and let \mathbb{Z}_{kh} denote the cyclic group of order kh . We let $g \in \mathbb{Z}_{kh}$ be a fixed choice of distinguished generator and let $\zeta = e^{\frac{2\pi i}{kh}} \in \mathbb{C}$ be a primitive kh^{th} root of unity.

Let $\mathbb{C}[V]$ denote the coordinate ring of polynomial functions $V \rightarrow \mathbb{C}$. If we fix a basis x_1, \dots, x_n of the dual vector space $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$, we may identify $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]$. The ring $\mathbb{C}[V]$ has a natural polynomial grading $\mathbb{C}[V] = \bigoplus_{d \geq 0} \mathbb{C}[V]_d$, so that $\mathbb{C}[V]_d$ can be thought of as polynomial functions $V \rightarrow \mathbb{C}$ of homogeneous degree d . We consider the graded $W \times \mathbb{Z}_{kh}$ -module on $\mathbb{C}[V]$ given as follows. The group W acts by linear substitutions. That is, we have $(w.f)(v) := f(w^{-1}.v)$. The distinguished generator g of the cyclic group \mathbb{Z}_{kh} scales by ζ^d in homogeneous degree d .

For a positive integer d , a *homogeneous system of parameters (h.s.o.p.) of degree d carrying V^** is a sequence of polynomials $\theta_1, \dots, \theta_n \in \mathbb{C}[V]$ (where $n = \dim(V)$ is the rank) such that the following conditions hold.

- We have that $\theta_1, \dots, \theta_n \in \mathbb{C}[V]_d$, i.e., the θ_i are homogeneous of degree d ,
- The zero locus cut out by $\theta_1 = \cdots = \theta_n = 0$ consists only of the origin $\{0\}$. Equivalently, the \mathbb{C} -vector space $\mathbb{C}[V]/(\Theta) := \mathbb{C}[V]/(\theta_1, \theta_2, \dots, \theta_n)$ is finite-dimensional.
- The \mathbb{C} -linear span $\text{span}_{\mathbb{C}}\{\theta_1, \dots, \theta_n\}$ is stable under the action of W .
- The \mathbb{C} -linear span $\text{span}_{\mathbb{C}}\{\theta_1, \dots, \theta_n\}$ is isomorphic to V^* as a $\mathbb{C}[W]$ -module.

In this paper we will be interested in h.s.o.p.'s of degree $d = kh + 1$ carrying V^* . These are uniformly known to exist by deep and subtle results from the theory of rational Cherednik algebras. In fact, Cherednik algebras can be used to produce an h.s.o.p. Θ of degree $kh + 1$ carrying V^* which is unique up to scaling.

More precisely, for any vector $v \in V$ there is a certain differential operator $D_v : \mathbb{C}[V] \rightarrow \mathbb{C}[V]$ called a *Dunkl operator* (see [2, Appendix] for its definition). It follows from Gordon's work on Cherednik algebras that there exists a W -equivariant linear map $\Theta : V^* \hookrightarrow \mathbb{C}[V]_{kh+1}$ whose image is annihilated by all the Dunkl operators $\{D_v : v \in V\}$ [13, 14]. Griffeth [15, Theorem 7.1]

proved that the map Θ is unique up to a nonzero scalar.¹ If x_1, \dots, x_n is any basis of V^* , then $\Theta(x_1), \dots, \Theta(x_n)$ gives a h.s.o.p. of degree $kh + 1$ carrying V^* .

We make no explicit use of Cherednik algebras in this paper, taking the existence of our h.s.o.p.'s as uniformly granted. We will refer to h.s.o.p.'s as in the above paragraph as “coming from Cherednik algebras”.

Observe that in the above situation of degree $d = kh + 1$, the ideal $(\Theta) \subset \mathbb{C}[V]$ is stable under the action of $W \times \mathbb{Z}_{kh}$, so that the quotient $\mathbb{C}[V]/(\Theta)$ has the structure of a $W \times \mathbb{Z}_{kh}$ -module. Bessis and Reiner [4] proved that the character $\chi : W \times \mathbb{Z}_{kh} \rightarrow \mathbb{C}$ of the representation $\mathbb{C}[V]/(\Theta)$ is given by the formula

$$(2.1) \quad \chi(w, g^d) = (kh + 1)^{\text{mult}_w(\zeta^d)}$$

Here $\text{mult}_w(\zeta^d)$ denotes the multiplicity of ζ^d as an eigenvalue in the action of w on V .

If $\theta_1, \dots, \theta_n \in \mathbb{C}[V]$ form a h.s.o.p. of degree $kh + 1$ carrying V^* , there exists an ordered basis x_1, \dots, x_n of V^* such that the linear map induced by the assignment $x_i \mapsto \theta_i$ is W -equivariant. The idea of an algebraic parking space comes from the inhomogeneous deformation given by replacing the ideal $(\theta_1, \dots, \theta_n)$ with the ideal $(\theta_1 - x_1, \dots, \theta_n - x_n)$. The following definition appears in [2] when $k = 1$ and [26] for general k .

Definition 2.2. *Let $\theta_1, \dots, \theta_n$ be a h.s.o.p. of degree $kh + 1$ carrying V^* . Fix an ordered basis x_1, x_2, \dots, x_n of V^* such that the linear map induced by $x_i \mapsto \theta_i$ is W -equivariant.*

The parking locus is the subscheme $V^\Theta(k)$ of V cut out by the ideal

$$(2.2) \quad (\Theta - \mathbf{x}) := (\theta_1 - x_1, \dots, \theta_n - x_n) \subset \mathbb{C}[V].$$

The k - W -algebraic parking space $\text{Park}_W^{\text{alg}}(k)$ is the associated quotient representation of $W \times \mathbb{Z}_{kh}$ given by

$$(2.3) \quad \text{Park}_W^{\text{alg}}(k) := \mathbb{C}[V]/(\Theta - \mathbf{x}).$$

In [2, Proposition 2.11] it is shown that we have a module isomorphism

$$(2.4) \quad \mathbb{C}[V]/(\Theta) \cong_{\mathbb{C}[W \times \mathbb{Z}_{kh}]} \mathbb{C}[V]/(\Theta - \mathbf{x}) = \text{Park}_W^{\text{alg}}(k),$$

so that our neither our choice of Θ nor our ideal deformation affect module structure.² As a result, the character $W \times \mathbb{Z}_{kh} \rightarrow \mathbb{C}$ of the algebraic parking space $\text{Park}_W^{\text{alg}}(k)$ is also given by the Equation 2.1.

Parking loci will be most important for us when the deformed ideal $(\Theta - \mathbf{x})$ is reduced. In this case, the parking locus $V^\Theta(k)$ is a set $V^\Theta(k) \subset V$ and may be identified with the set of fixed points of the map $\Theta : V \rightarrow V$ which sends a point with coordinates (x_1, \dots, x_n) to a point with coordinates $(\theta_1, \dots, \theta_n)$. This fixed point perspective explains the superscript notation in $V^\Theta(k)$. The set $V^\Theta(k)$ carries a permutation action of $W \times \mathbb{Z}_{kh}$, where W acts by linear substitutions and the distinguished generator $g \in \mathbb{Z}_{kh}$ scales by ζ .

If the parking locus $V^\Theta(k)$ is reduced, we get a canonical identification $V^\Theta(k) \cong_{\mathbb{C}[W \times \mathbb{Z}_{kh}]} \text{Park}_W^{\text{alg}}(k)$.³ Therefore, the $W \times \mathbb{Z}_{kh}$ -set $V^\Theta(k)$ has permutation character given by Equation 2.1. In particular, the parking locus $V^\Theta(k)$ contains $(kh + 1)^n$ points. The following result of Etingof shows that there exists a choice of Θ such that $V^\Theta(k)$ is reduced.

¹The case of the symmetric group $\mathfrak{S}_n = G(1, 1, n)$ is not included in [15, Theorem 7.1], but can be deduced from its statement.

²While the choice of Θ could *a priori* affect ring structure, we have not had occasion to use the ring structures of $\mathbb{C}[V]/(\Theta)$ or $\mathbb{C}[V]/(\Theta - \mathbf{x})$ in our work. For cleanliness of notation, we drop reference to Θ in the algebraic parking space $\text{Park}_W^{\text{alg}}(k)$. At any rate, the parking loci $V^\Theta(k)$ will be the focus of this paper.

³Here we are using the fact the W is *real*, so that its reflection representation is self-dual.

Theorem 2.3. (*Etingof, see [2, Appendix]*) *Let Θ_0 be the h.s.o.p. of degree $kh + 1$ carrying V^* coming from Cherednik algebras. The parking locus $V^{\Theta_0}(k)$ is reduced, and so consists of $(kh + 1)^n$ distinct points in V .*

Etingof's argument uses the fact that the image of the map corresponding to Θ_0 is annihilated by all Dunkl operators. The Cartan-theoretic definition of the Dunkl operators can be used, together with a Schur's Lemma argument, to prove the reducedness of $V^{\Theta_0}(k)$. Unfortunately, the characterization of Θ_0 in terms of Dunkl operators has not yet proved sufficient to understand $V^{\Theta_0}(k)$ explicitly enough so that its $W \times \mathbb{Z}_{kh}$ -structure can be connected with noncrossing parking functions.

Remark 2.4. *Let us motivate the use of $V^{\Theta}(k)$ as a model for parking functions.*

Assume that W is crystallographic. The action of W on V stabilizes the root lattice $Q = \mathbb{Z}[\Phi] \subset V$. Consider the dilation $(kh + 1)Q$ of the lattice Q . The group W acts on the finite torus $Q/(kh + 1)Q \cong (\mathbb{Z}_{kh+1})^n$. The use of finite tori to uniformly model parking functions in crystallographic type goes back to the origins of parking functions in algebraic combinatorics [16].

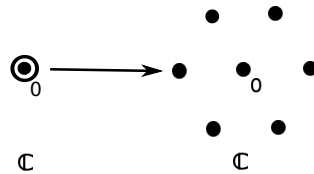
Outside of crystallographic type, there is no root lattice Q and this construction breaks down. When $V^{\Theta}(k)$ is reduced, we can identify the finite set $V^{\Theta}(k) \subset V$ as a finite torus-like object outside of crystallographic type. The construction of $V^{\Theta}(k)$ even applies to well-generated complex reflection groups, although we don't pursue this here.

Even inside crystallographic type, the locus $V^{\Theta}(k)$ has a significant advantage over the finite torus $Q/(kh + 1)Q$: it carries a natural action of not just W , but the product group $W \times \mathbb{Z}_{kh}$. This additional cyclic group action is closely related to the action of rotation on noncrossing partitions.

In order to think about continuously varying families of h.s.o.p.'s, it will be useful to think of h.s.o.p.'s in terms of polynomial maps $V \rightarrow V$. Fix a basis x_1, \dots, x_n of the dual space V^* . For any positive integer d , the affine space $\text{Hom}_{\mathbb{C}}(V^*, \mathbb{C}[V]_d)$ parametrizes the collection of all degree d homogeneous polynomial maps $\Theta : V \rightarrow V$, where we have $x_i(\Theta(v)) := \Theta(x_i)(v)$ for all $v \in V$ and $1 \leq i \leq n$. In other words, we have that $(\theta_1, \dots, \theta_n) := (\Theta(x_1), \dots, \Theta(x_n))$ are the coordinate functions of Θ with respect to x_1, \dots, x_n .

The group W acts on both V^* and $\mathbb{C}[V]_d$. Under the above setup, the equivariant affine space $\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_d)$ parametrizes the collection of all degree d homogeneous polynomial maps $\Theta : V \rightarrow V$ which are W -equivariant: $\Theta(w.v) = w.\Theta(v)$ for all $w \in W$ and $v \in V$. An h.s.o.p. of degree d carrying V^* is nothing more than an element $\Theta \in \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_d)$ whose associated function $\Theta : V \rightarrow V$ satisfies $\Theta^{-1}(0) = \{0\}$.

Example 2.5. *Let us consider the case of rank 1. We have the identifications $W = \{\pm 1\}$, $V = \mathbb{C}$, and $\mathbb{C}[V] = \mathbb{C}[x]$. Up to a choice of scalar, the unique h.s.o.p. of degree $kh + 1 = 2k + 1$ is $\theta_1 = x^{2k+1}$ and the W -equivariant map $V^* \rightarrow \mathbb{C}[V]_{2k+1}$ is given by $x \mapsto x^{kh+1}$. The ideal $(\Theta) = (x^{2k+1}) \subset \mathbb{C}[x]$ corresponding to a fat point at the origin in \mathbb{C} of multiplicity $2k + 1$. The parking locus $V^{\Theta}(k)$ corresponds to the deformed ideal $(\Theta - \mathbf{x}) = (x^{2k+1} - x)$, so we may identify $V^{\Theta}(k)$ with the 'blown apart' locus of $2k + 1$ points $V^{\Theta}(k) = \{1, \zeta, \zeta^2, \dots, \zeta^{2k-1}, 0\}$, where $\zeta = e^{\frac{\pi i}{k}}$. This process shown below in the case $k = 3$. The generator $g \in \mathbb{Z}_{kh} = \mathbb{Z}_{2k}$ scales by ζ and W acts by ± 1 .*



In types BCDI, h.s.o.p.'s of degree $kh + 1$ (or more generally of any odd degree) carrying V^ can be obtained by taking powers $x_1^{kh+1}, \dots, x_n^{kn+1}$ of the coordinate functions on the standard models of*

the reflection representations. The case of the symmetric group is much harder, essentially because the standard action of \mathfrak{S}_n on \mathbb{C}^n fails to be irreducible. An inductively constructed h.s.o.p. which depends on the prime factorization of n is due to Kraft and can be found in [16]. Chmutova and Etingof [8] have an explicit h.s.o.p. involving formal power series. We will not use any explicit h.s.o.p.'s in this paper.

2.4. k - W -noncrossing partitions. In [2] a certain $W \times \mathbb{Z}_h$ -set Park_W^{NC} called the set of W -noncrossing parking functions was constructed. In [26] this definition was extended to give a Fuss analog in the form of a $W \times \mathbb{Z}_{kh}$ -set $\text{Park}_W^{NC}(k)$ of k - W -noncrossing parking functions for any positive integer k (so that we have the specialization $\text{Park}_W^{NC} = \text{Park}_W^{NC}(1)$). We recall the definition of $\text{Park}_W^{NC}(k)$ and give its combinatorial model in type A.

For any $w \in W$, denote by V^w the corresponding fixed space $V^w := \{v \in V : w.v = v\}$. Given $X \subseteq V$, denote by $W_X := \{w \in W : w.x = x \text{ for all } x \in X\}$ the subgroup of W which fixes X pointwise. The subgroups W_X are known as *parabolic subgroups* of W .

Recall that $\text{Cox}(W)$ is the Coxeter arrangement in V attached to W . Let \mathcal{L} denote the intersection lattice of this arrangement. Subspaces $X \in \mathcal{L}$ are called *flats*.

Given $w \in W$, the *reflection length* $\ell_T(w)$ is the minimum number l such that we can write $w = t_1 t_2 \cdots t_l$ with $t_1, t_2, \dots, t_l \in T$. *Absolute order* is the partial order \leq_T on W defined by $u \leq_T w$ if and only if we have $\ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$.

The absolute order \leq_T on W has a unique minimal element given by the identity $e \in W$, but usually has many maximal elements. The Coxeter elements of W are all maximal. We let $[e, c]_T := \{w \in W : e \leq_T w \leq_T c\}$ denote the absolute order interval between e and c . The group elements in the interval $NC(W) := [e, c]_T$ are called *noncrossing*.

For any group element $w \in W$, the fixed space V^w is in \mathcal{L} . The map $W \rightarrow \mathcal{L}$ given by $w \mapsto V^w$ restricts to an injection $NC(W) \hookrightarrow \mathcal{L}$. Flats in the image of this injection are called *noncrossing*.

Following Armstrong [1], we define the *k - W -noncrossing partitions* $NC^k(W)$ to be the set of all k -element multichains $(w_1 \leq_T \cdots \leq_T w_k)$ in the poset $NC(W)$ of W -noncrossing partitions. These multichains were also considered by Chapoton [7]. Applying the fixed space map, we arrive at the notion of a *noncrossing k -flat*, which is a descending multichain $(X_1 \supseteq \cdots \supseteq X_k)$ of noncrossing flats in \mathcal{L} .

The set $NC^k(W)$ can be interpreted in terms of factorizations of the distinguished Coxeter element c . A sequence $(w_0, w_1, \dots, w_k) \in W^{k+1}$ is called an *ℓ_T -additive factorization of c of length $k+1$* if $w_0 w_1 \cdots w_k = c$ and $\ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_k) = \ell_T(c) = n$. We let $NC_k(W)$ denote the set of ℓ_T -additive factorizations of c of length $k+1$. The following ‘difference and sum’ maps ∂ and \int are mutually inverse bijections between $NC^k(W)$ and $NC_k(W)$.

$$\begin{aligned} \partial : NC^k(W) &\longrightarrow NC_k(W) \\ \partial : (w_1, w_2, \dots, w_k) &\mapsto (w_1, w_1^{-1}w_2, \dots, w_{k-1}^{-1}w_k, w_k^{-1}c) \\ \int : NC_k(W) &\longrightarrow NC^k(W) \\ \int : (w_0, w_1, \dots, w_k) &\mapsto (w_0, w_0w_1, \dots, w_0w_1, w_{k-1}) \end{aligned}$$

The cyclic group $\mathbb{Z}_{kh} = \langle g \rangle$ acts on $NC_k(W)$ by $g.(w_0, w_1, \dots, w_k) := (v, cw_k c^{-1}, w_1, w_2, \dots, w_{k-1})$, where $v = (cw_k c^{-1})w_0(cw_k c^{-1})^{-1}$ (see [1]). By transferring structure through the bijection \int , we get an action of \mathbb{Z}_{kh} on the set $NC^k(W)$ of k - W -noncrossing partitions. By taking fixed spaces, we also get an action $(X_1 \supseteq \cdots \supseteq X_k) \mapsto g.(X_1 \supseteq \cdots \supseteq X_k)$ on the set of noncrossing k -flats. This action is called *generalized rotation*.

Example 2.6. We will only consider W -noncrossing partitions and W -noncrossing parking functions in any specificity when $W = \mathfrak{S}_n$ is the symmetric group. Let us review the relevant combinatorics of noncrossing partitions in type A.

The Coxeter arrangement for $W = \mathfrak{S}_n$ is the braid arrangement $\{x_i - x_j = 0 : 1 \leq i < j \leq n\}$. We may identify flats $X \in \mathcal{L}$ with set partitions π of $[n]$ by letting $i \sim j$ if and only if the coordinate equality $x_i = x_j$ holds on X . When $c = (1, 2, \dots, n)$, a flat X is noncrossing if and only if the corresponding set partition π of $[n]$ is noncrossing in the sense that the blocks of π do not cross when drawn on a disc with boundary labelled clockwise by $1, 2, \dots, n$.

Noncrossing k -flats ($X_1 \supseteq \dots \supseteq X_k$) may be identified with noncrossing set partitions π of $[kn]$ which are k -divisible in the sense that every block of π has size divisible by k . Under this identification, generalized rotation is the usual rotation action on noncrossing set partitions.

2.5. Noncrossing parking functions. Our combinatorial model of parking functions is given by the following set of equivalence classes, which appeared in [2] when $k = 1$ and in [26] for general k .

Definition 2.7. A k - W -noncrossing parking function is an equivalence class in

$$(2.5) \quad \{(w, X_1 \supseteq \dots \supseteq X_k) : X_1 \supseteq \dots \supseteq X_k \text{ a noncrossing } k\text{-flat}\} / \sim,$$

where $(w, X_1 \supseteq \dots \supseteq X_k) \sim (w', X'_1 \supseteq \dots \supseteq X'_k)$ if $X_i = X'_i$ for all i and we have the coset equality $wW_{X_1} = w'W_{X_1}$.

The set of k - W -noncrossing parking functions is denoted $\text{Park}_W^{NC}(k)$.

We use square brackets to denote equivalence classes, so that $[w, X_1 \supseteq \dots \supseteq X_k]$ is the k - W -noncrossing parking function containing $(w, X_1 \supseteq \dots \supseteq X_k)$. By [26, Proposition 3.2], the rule

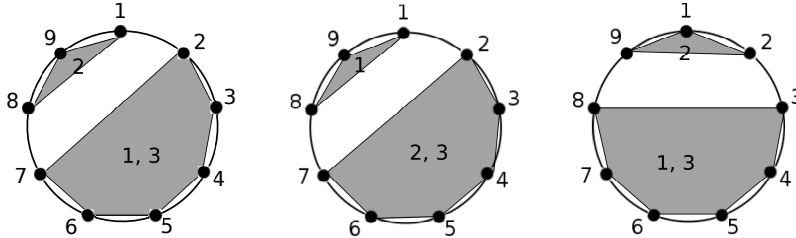
$$(2.6) \quad (v, g) \cdot [w, X_1 \supseteq \dots \supseteq X_k] := [vwu_k c^{-1}, g \cdot (X_1 \supseteq \dots \supseteq X_k)]$$

induces a well defined action of the group $W \times \mathbb{Z}_{kh}$ on $\text{Park}_W^{NC}(k)$, where $u_k \in W$ is the unique noncrossing group element such that $V^{u_k} = X_k$.

Example 2.8. When $W = \mathfrak{S}_n$, we can visualize noncrossing parking functions using noncrossing partitions. There is a bijection ∇ (see [26] for its definition) between $\text{Park}_{\mathfrak{S}_n}^{NC}(k)$ and pairs (π, f) where

- π is a k -divisible noncrossing partition of $[kn]$,
- $f : B \mapsto f(B)$ is a labeling of the blocks of π with subsets of $[n]$,
- if B is a block of π , we have that $|f(B)| = \frac{|B|}{k}$, and
- we have $[n] = \bigsqcup_{B \in \pi} f(B)$.

When $n = k = 3$, three elements of $\text{Park}_{\mathfrak{S}_3}^{NC}(3)$ are shown below. The element on the left corresponds to the pair (π, f) where $\pi = \{\{1, 8, 9\}, \{2, 3, 4, 5, 6, 7\}\}$, $f(\{1, 8, 9\}) = \{2\}$, and $f(\{2, 3, 4, 5, 6, 7\}) = \{1, 3\}$.



The bijection given in [26] makes the $W \times \mathbb{Z}_{kh} = \mathfrak{S}_n \times \mathbb{Z}_{kn}$ action easy to visualize. The symmetric group \mathfrak{S}_n acts by permuting labels, leaving the noncrossing partition π fixed. The center parking function above is the image of the left parking function under $(1, 2) \in \mathfrak{S}_3$. The distinguished generator $g \in \mathbb{Z}_{kn}$ acts by clockwise rotation. The right parking function above is the image of the left parking function under $g \in \mathbb{Z}_9$.

We remark that the study of labeled noncrossing partitions (π, f) goes back to a 1980 paper of Edelman [10, Section 5]. In the case $k = 1$, our pairs (π, f) are what Edelman calls ‘non-crossing 2-partitions’. Edelman defines a partial order T_n^2 on n.c. 2-partitions and uses Lagrange Inversion

to prove that multichains in this partial order are counted by the formula $(kn+1)^{n-1}$ [10, Theorem 5.3]. The bijection ∇ in [26] (which is a parking function enrichment of Armstrong's map ∇ given in [1]) translates k -element multichains in T_n^2 to ordered pairs (π, f) for general k as above.

2.6. The Main Conjecture. The Weak form of the Main Conjecture gives a character formula for our combinatorial model of parking functions.

Weak Conjecture. Let $\chi : W \times \mathbb{Z}_{kh} \rightarrow \mathbb{C}$ be the permutation character of the $W \times \mathbb{Z}_{kh}$ -set $\text{Park}_W^{NC}(k)$. For any $w \in W$ and $d \geq 0$ we have that

$$(2.7) \quad \chi(w, g^d) = (kh+1)^{\text{mult}_w(\zeta^d)},$$

where $\zeta = e^{\frac{2\pi i}{kh}}$ is a primitive $(kh)^{\text{th}}$ root-of-unity and $\text{mult}_w(\zeta^d)$ is the multiplicity of ζ^d as an eigenvalue in the action of w on V .

The Weak Conjecture uniformly implies a number of facts in W -Catalan theory which are at present only understood in a case-by-case fashion. In particular, for any W for which the Weak Conjecture holds, we can *uniformly* prove the following facts.

- (1) (Fuss-Catalan Count) The number $|NC^k(W)|$ of k - W -noncrossing partitions is the W -Fuss-Catalan number $\text{Cat}^k(W) := \prod_{i=1}^n \frac{kh+d_i}{d_i}$, where d_1, \dots, d_n are the invariant degrees of W .
- (2) (Fuss-Catalan CSP) The triple $(NC^k(W), \mathbb{Z}_{kh}, \text{Cat}_q^k(W))$ exhibits the *cyclic sieving phenomenon* (see [25]), where the cyclic group \mathbb{Z}_{kh} acts on the set $NC^k(W)$ by generalized rotation and $\text{Cat}_q^k(W) := \prod_{i=1}^n \frac{1-q^{kh+d_i}}{1-q^{d_i}}$ is the q - W -Fuss-Catalan number. This means that the number of elements in $NC^k(W)$ fixed by g^d equals the polynomial evaluation $[\text{Cat}_q^k(W)]_{q=\zeta^d}$.
- (3) (Kreweras Coincidence) Assume W has crystallographic type. For any flat $X \in \mathcal{L}$, the number of noncrossing flats in the orbit $W.X$ of X under the action of W equals the number of nonnesting flats in this orbit.⁴

Fact 1 above is a specialization of Fact 2 at $q = 1$. When $k = 1$, Bessis and Reiner [24, 4] proved Facts 1 and 2 by combinatorial models in the infinite families ABCDI and computer checks in the exceptional types EFH. Fact 3 was used by Bessis and Reiner to prove Fact 2 [4]. For general $k \geq 1$, the Fuss-Catalan Count of multichains in the absolute order interval $[e, c]_T$ was performed by Chapoton [7]. The cyclic sieving result for general k is due to Krattenthaler-Müller [19, 20] and Kim [17]. At present, Facts 1-3 are only understood in a case-by-case fashion. The following result motivates the Weak Conjecture.

Proposition 2.9. *The Weak Conjecture uniformly implies Facts 1-3 whenever it is true.*

Proof. For Facts 1 and 2, this is described in [2] for $k = 1$ and [26] for $k \geq 1$. For Fact 3, for any flat X one considers the permutation action of W on the parabolic cosets W/W_X . It is well known that the characters $\{\psi_X : W \rightarrow \mathbb{C}\}$ of these actions are linearly independent as X varies over a collection of W -orbit representatives in the intersection lattice \mathcal{L} . Ignoring the \mathbb{Z}_h -action on $\text{Park}_W^{NC}(1)$, we get that the inner product $\langle \chi \downarrow_W, \psi_X \rangle_W$ of the W -character $\chi \downarrow_X$ of $\text{Park}_W^{NC}(1)$ with ψ_X is the number of W -noncrossing flats in the orbit of X . On the other hand, the finite torus $Q/(h+1)Q$ is uniformly known to have W -character as in the Weak Conjecture [16]. The W -orbits in $Q/(h+1)Q$ biject with W -nonnesting flats [6, 28], and an orbit corresponding to a nonnesting flat X contributes ψ_X to the corresponding character. \square

The author thanks Vic Reiner for pointing out the proof of Proposition 2.9 shown above. Further uniform ramifications of the Weak Conjecture concerning Kirkman and Narayana numbers can be

⁴Following Postnikov (see [23, Remark 2], a flat $X \in \mathcal{L}$ is *nonnesting* if it is a hyperplane intersection corresponding to an antichain in the positive root poset Φ^+ .

found in [2]. In light of the above discussion, a uniform proof of the Weak Conjecture would be highly desirable. One approach for doing so would be to give a uniform proof of the following Intermediate Conjecture, which relates noncrossing parking functions to h.s.o.p.'s.

Intermediate Conjecture. *There exists a h.s.o.p. $\Theta \in \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ such that the parking locus $V^\Theta(k)$ is reduced and there is a $W \times \mathbb{Z}_{kh}$ -equivariant bijection of sets*

$$V^\Theta(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k).$$

Given a h.s.o.p. Θ satisfying the conditions of the Intermediate Conjecture, we can use the isomorphism $V^\Theta(k) \cong_{\mathbb{C}[W \times \mathbb{Z}_{kh}]} \text{Park}_W^{alg}(k)$ to deduce the Weak Conjecture uniformly. We remark that the conclusion of the Intermediate Conjecture is *a priori* stronger than that of the Weak Conjecture. While the Weak Conjecture would guarantee a *linear* isomorphism of $\mathbb{C}[W \times \mathbb{Z}_{kh}]$ -modules $V^\Theta(k) \cong_{\mathbb{C}[W \times \mathbb{Z}_{kh}]} \text{Park}_W^{NC}(k)$, it does not guarantee the stronger property of a $W \times \mathbb{Z}_{kh}$ -set bijection $V^\Theta(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k)$.⁵

In this paper we will prove the Intermediate Conjecture in type A. The main obstruction to proving the Intermediate Conjecture in type A has been the relative complexity of the h.s.o.p.'s making the locus $V^\Theta(k)$ difficult to analyze. In our proof, we will not calculate this locus explicitly, but instead compare stabilizers of points within $V^\Theta(k)$ and $\text{Park}_W^{NC}(k)$ and use sieve techniques to deduce the relevant bijection.

The Strong Conjecture asserts that the conclusion of the Intermediate Conjecture holds for any h.s.o.p. $\Theta \in \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$.

Strong Conjecture. *For any element $\Theta \in \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ such that Θ is an h.s.o.p., the parking locus $V^\Theta(k)$ is reduced and there is a $W \times \mathbb{Z}_{kh}$ -equivariant bijection of sets*

$$V^\Theta(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k).$$

Our generic analog of the Strong Conjecture is as follows.

Generic Strong Conjecture. *Let $\mathcal{R} \subset \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ denote the set of polynomial maps $\Theta \in \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ such that Θ is a h.s.o.p. and $V^\Theta(k)$ is reduced.*

- *For any $\Theta \in \mathcal{R}$ we have a $W \times \mathbb{Z}_{kh}$ -equivariant bijection of sets*

$$V^\Theta(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k).$$

- *There is a nonempty Zariski open set $\mathcal{U} \subset \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ satisfying $\mathcal{U} \subseteq \mathcal{R}$.*

Aside from the assertion about \mathcal{U} , the Strong Conjecture clearly implies the Generic Strong Conjecture. Moreover, the Generic Strong Conjecture implies the Intermediate Conjecture. We will show that the Generic Strong and Intermediate Conjectures are, in fact, equivalent. This will prove the Generic Strong Conjecture in all infinite families ABCDI. We will also show uniformly that there *always* exists a nonempty Zariski open \mathcal{U} with $\mathcal{U} \subseteq \mathcal{R}$.

Remark 2.10. *The reader may question the usefulness of the Strong Conjecture. Given Etingof's Theorem 2.3 and the fact that proving $V^\Theta(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k)$ for just one h.s.o.p. Θ would yield the desired uniform proofs of Facts 1-3, why not take Θ to be the h.s.o.p. coming from Cherednik algebras? And, given the difficulty of constructing relevant h.s.o.p.'s uniformly, do we know that we really have more freedom in our choice of Θ ?*

⁵If we ignore the cyclic group action, the linear independence of the characters ψ_X in the proof of Proposition 2.9 shows that the linear isomorphism $V^\Theta(k) \cong_{\mathbb{C}[W]} \text{Park}_W^{NC}(k)$ uniformly implies the combinatorial isomorphism $V^\Theta(k) \cong_W \text{Park}_W^{NC}(k)$. The author is unaware of a similar linear independence result for $W \times \mathbb{Z}_{kh}$ -characters. At any rate, an explicit $W \times \mathbb{Z}_{kh}$ -equivariant bijection between $V^\Theta(k)$ and $\text{Park}_W^{NC}(k)$ would be desirable for combinatorial understanding.

To compute the locus $V^\Theta(k)$, we need to solve a system of polynomial equations arising from the h.s.o.p. Θ . When Θ is only understood as an element of the common kernel of Dunkl operators, solving such a system seems difficult. The Strong Conjecture represents the hope that parking functions can be understood without recourse to Cherednik algebras.

The Zariski openness of \mathcal{U} in the Generalized Strong Conjecture can be interpreted as saying that the dimension of the “parameter space” of relevant Θ for the Strong Conjecture is measured by the dimension of the vector space $\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$. The following result shows that these dimensions can be large, meaning that there are significantly more choices of Θ than the one coming from Cherednik algebras.

In order to state the dimension of $\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$, let us introduce some notation. If $\mu = (\mu_1 \geq \dots \geq \mu_n)$ is a weakly decreasing sequence of nonnegative integers, let $\delta(\mu)$ be the number of distinct entries in μ , less one. The following result shows that the dimension of \mathcal{U} in the Generalized Strong Conjecture can be much greater than one.

Proposition 2.11. *The dimension of the \mathbb{C} -vector space $\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ in types ABCDI is as follows. In the formulas below, the μ_i and ν_i are nonnegative integers.*

- When $W = \mathfrak{S}_n$, we have $h = n$ and

$$(2.8) \quad \dim(\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})) = \sum_{\substack{\mu=(\mu_1 \geq \dots \geq \mu_n) \\ \sum_i \mu_i = kn+1}} \delta(\mu) - \sum_{\substack{\nu=(\nu_1 \geq \dots \geq \nu_n) \\ \sum_i \nu_i = kn}} \delta(\nu).$$

- When $W = W(B_n) = W(C_n)$, we have $h = 2n$ and

$$(2.9) \quad \dim(\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})) = \# \left\{ (\mu_1 \geq \dots \geq \mu_n) : \sum \mu_i = 2nk + 1, \text{ exactly one } \mu_i \text{ odd} \right\}.$$

- When $W = W(D_n)$, we have $h = 2n - 2$ and

$$(2.10) \quad \dim(\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})) = \# \left\{ (\mu_1 \geq \dots \geq \mu_n) : \sum \mu_i = (2n - 2)k + 1, \text{ exactly one } \mu_i \text{ odd} \right\}.$$

- When $W = W(I_2(m))$, we have $h = m$ and

$$(2.11) \quad \dim(\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})) = k + 1.$$

Proof. While these results are somewhat standard, we perform the relevant calculations for the convenience of the reader.

First consider the case $W = \mathfrak{S}_n$ for $n > 1$. The defining representation \mathbb{C}^n decomposes as $\mathbb{C}^n = V \oplus \mathbf{1}_{\mathfrak{S}_n}$. For any degree d , we have the following identifications of \mathfrak{S}_n -modules: $\text{Sym}^d(\mathbb{C}^n) = \text{Sym}^d(V \oplus \mathbf{1}_{\mathfrak{S}_n}) = \bigoplus_{i=0}^d \text{Sym}^i(V) \otimes \text{Sym}^{d-i}(\mathbf{1}_{\mathfrak{S}_n}) = \bigoplus_{i=0}^d \text{Sym}^i(V)$. Since this is true for any d , we conclude that $\text{Sym}^d(V) \oplus \text{Sym}^{d-1}(\mathbb{C}^n) = \text{Sym}^d(\mathbb{C}^n)$. This allows us to compute the desired dimension by finding the multiplicity of V in $\text{Sym}^d(\mathbb{C}^n)$ and $\text{Sym}^{d-1}(\mathbb{C}^n)$ and then subtracting.

We can identify $\text{Sym}^d(\mathbb{C}^n)$ with the action of \mathfrak{S}_n on degree d monomials in the variables x_1, \dots, x_n . A system of orbit representatives is given by $\{x_1^{\mu_1} \dots x_n^{\mu_n}\}$, where $\mu_1 \geq \dots \geq \mu_n$ and $\sum \mu_i = d$. We may identify V with the \mathfrak{S}_n -irreducible S^λ corresponding to the partition $\lambda = (n-1, 1) \vdash n$. Young’s Rule tells us that for $\mu = (\mu_1 \geq \dots \geq \mu_n)$, the multiplicity of V in the \mathfrak{S}_n -module generated by $x_1^{\mu_1} \dots x_n^{\mu_n}$ is $\delta(\mu)$. This completes the case $W = \mathfrak{S}_n$.

Suppose that W has type B_n/C_n (and $n \geq 2$) or D_n (and $n \geq 4$). We may identify V with the defining action of W on \mathbb{C}^n . As before, we identify $\text{Sym}^d(\mathbb{C}^n)$ with the (signed) permutation action of W on the collection of degree d monomials in the variables x_1, \dots, x_n . Given $\mu = (\mu_1 \geq \dots \geq \mu_n)$ with $\sum \mu_i = d$, we get a corresponding submodule M^μ generated by $x_1^{\mu_1} \dots x_n^{\mu_n}$.

We claim that the multiplicity of V in M^μ equals either one or zero, according to whether μ contains a unique odd entry or not. To see why this is the case, consider the standard embeddings $W(B_{n-1}) \subset W(B_n)$ and $W(D_{n-1}) \subset W(D_n)$. Any copy of V inside M^μ must be an

n -dimensional submodule containing an $(n-1)$ -dimensional subspace $V' \subset V$ on which the subgroups $W(B_{n-1})/W(D_{n-1})$ act trivially. If μ contains more than one odd part, it is impossible to find such a V' . If μ contains no odd parts, the diagonal subgroup $W \cap \text{diag}(\pm 1, \dots, \pm 1)$ acts trivially on M^μ , so that M^μ contains no copy of V . In μ contains precisely one odd part μ_i , the unique copy of V inside M^μ is generated by $x_i^{\mu_i} \sum_{w \in \mathfrak{S}_{[n]-\{i\}}} x_1^{\mu_{w(1)}} \cdots \widehat{x_i^{\mu_{w(i)}}} \cdots x_n^{\mu_{w(n)}} \in M^\mu$.

Finally, suppose that $W = I_2(m)$. We consider the generating set of the dihedral group W given by the two matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. The $k+1$ copies of the reflection representation V sitting inside the space $\mathbb{C}[V]_{km+1} = \mathbb{C}[x, y]_{km+1}$ are spanned by the sets

$$\{x^{km+1}, y^{km+1}\}, \{x^{(k-1)m+1}y^m, x^m y^{(k-1)m+1}\}, \dots, \{xy^{km}, x^{km}y\}.$$

□

3. PARKING STABILIZERS

In this section and the next, we will prove the Intermediate Conjecture in type A. The results and proofs in this section are uniform; we specialize to type A in the next section.

In order to prove the Intermediate Conjecture, we need to prove an isomorphism of $W \times \mathbb{Z}_{kh}$ -sets. For any finite group G , to prove that a given pair of finite-dimensional $\mathbb{C}[G]$ -modules are isomorphic, it is enough to show that their characters coincide. On the other hand, for general finite groups G there can be two finite G -sets \mathcal{S} and \mathcal{T} with the same (permutation) character such that there is no G -equivariant bijection $\mathcal{S} \cong_G \mathcal{T}$. To prove our $W \times \mathbb{Z}_{kh}$ -set isomorphisms, we will rely on the following basic sieve-type result.

Lemma 3.1. *Let G be a finite group and let \mathcal{S} and \mathcal{T} be finite G -sets. Suppose that for every subgroup $H \leq G$ which arises as the stabilizer of an element of \mathcal{S} or \mathcal{T} , the corresponding fixed point sets have the same cardinality:*

$$|\mathcal{S}^H| = |\mathcal{T}^H|.$$

Then there is a G -equivariant bijection $\mathcal{S} \cong_G \mathcal{T}$.

Proof. Consider the poset P of subgroups of G which arise as stabilizers of elements of \mathcal{S} or \mathcal{T} , ordered by inclusion. For any subgroup $H \in P$, the hypothesis $|\mathcal{S}^H| = |\mathcal{T}^H|$ may be rewritten as

$$(3.1) \quad \sum_{H \leq_P K} |\{s \in \mathcal{S} : \text{Stab}_G(s) = K\}| = \sum_{H \leq_P K} |\{t \in \mathcal{T} : \text{Stab}_G(t) = K\}|.$$

Since P is a finite poset and Equation 3.1 holds for *all* $H \in P$, we have

$$(3.2) \quad |\{s \in \mathcal{S} : \text{Stab}_G(s) = H\}| = |\{t \in \mathcal{T} : \text{Stab}_G(t) = H\}|$$

for all subgroups $H \in P$.

Taking $H = \{e\}$, we get that $|\mathcal{S}| = |\mathcal{T}|$. We argue by induction on this common cardinality. Choose $s_0 \in \mathcal{S}$ arbitrarily. By Equation 3.2, there exists $t_0 \in \mathcal{T}$ such that $\text{Stab}_G(s_0) = \text{Stab}_G(t_0)$. Extend the assignment $s_0 \mapsto t_0$ in the unique way to get a G -equivariant bijection between the orbits $G.s_0 \xrightarrow{\sim} G.t_0$. On the other hand, since

$$(3.3) \quad \text{Stab}_G(g.s_0) = g\text{Stab}_G(s_0)g^{-1} = g\text{Stab}_G(t_0)g^{-1} = \text{Stab}_G(g.t_0)$$

for all $g \in G$, the hypothesis of the lemma continues to hold when one replaces \mathcal{S} with $\mathcal{S} - G.s_0$ and \mathcal{T} with $\mathcal{T} - G.t_0$. The lemma follows from induction. □

The G -sets we will apply Lemma 3.1 to will be the $W \times \mathbb{Z}_{kh}$ -sets $\text{Park}_W^{NC}(k)$ and $V^\Theta(k)$. In order to apply Lemma 3.1 effectively, we will need to characterize the subgroups $H \leq W \times \mathbb{Z}_{kh}$ which arise as stabilizers in the actions on $\text{Park}_W^{NC}(k)$ or $V^\Theta(k)$ compute the fixed set sizes $|\text{Park}_W^{NC}(k)^H|$ and $|V^\Theta(k)^H|$. The fundamental enumerative result which allows us to count $V^\Theta(k)^H$ is as follows.

Lemma 3.2. *Let $\Theta : V \rightarrow V$ be a h.s.o.p. of degree $kh + 1$ carrying V^* such that the parking locus $V^\Theta(k)$ is reduced. Let $X \subseteq V$ be any subspace which is stabilized by Θ . The intersection $X \cap V^\Theta(k)$ has precisely $(kh + 1)^{\dim(X)}$ points.*

Proof. In fact, this is true even if Θ does not commute with the action of W . By assumption, we can restrict Θ to get a polynomial map $\Theta|_X : X \rightarrow X$ of homogeneous degree $kh + 1$. By Bézout's Theorem and the fact that $\Theta|_X^{-1}(0) = \{0\}$, we get that $\Theta|_X$ has $(kh + 1)^{\dim(X)}$ fixed points counting multiplicity. The multiplicities of all of these fixed points must equal 1 because the locus $V^\Theta(k)$ is reduced. \square

In order to apply Lemma 3.2, we will need to find some interesting subspaces $X \subseteq V$ which are stabilized by Θ . The subspaces we will consider will be intersections of subspaces given in the following lemma. For $w \in W$ and $\xi \in \mathbb{C}$, let $E(w, \xi) := \{v \in V : w.v = \xi v\}$ be the corresponding eigenspace in the action of w on V . In particular, we have $E(w, 1) = V^w$.

Lemma 3.3. *Let $\xi \in \mathbb{C}$ be a complex number satisfying $\xi^{kh} = 1$ and let Θ be any h.s.o.p. of degree $kh + 1$ carrying V^* . For any $w \in W$, the eigenspace $E(w, \xi)$ is stabilized by Θ .*

Proof. Let $v \in E(w, \xi)$. We compute

$$(3.4) \quad w.\Theta(v) = \Theta(w.v) = \Theta(\xi v) = \xi^{kh+1}\Theta(v) = \xi\Theta(v),$$

where the first equality uses the fact that Θ commutes with the action of W , the second uses the fact that $v \in E(w, \xi)$, the third uses the fact that Θ is homogeneous of degree $kh + 1$, and the fourth uses the fact that $\xi^{kh} = 1$. We conclude that $\Theta(v) \in E(w, \xi)$. \square

When $\xi = 1$, Lemma 3.2 implies that the flats X of the intersection lattice \mathcal{L} are stable under the action of Θ . Alex Miller studied the poset of ξ -eigenspaces $E(w, \xi)$ of elements w of a complex reflection group W for a fixed $\xi \neq 1$, ordered by reverse inclusion [22]. In this paper we will consider “mixed” subspaces which are intersections of the form $X \cap E(w, \xi)$, where $X \in \mathcal{L}$ and $\xi^{kh} = 1$. The study of *arbitrary* intersections of eigenspaces (corresponding to possibly different roots of unity ξ) could yield interesting combinatorics.

In order to apply Lemma 3.1, we will need to determine which subgroups $H \leq W \times \mathbb{Z}_{kh}$ arise as stabilizers of elements of $\text{Park}_W^{NC}(k)$ or $V^\Theta(k)$. In particular, this should be the same collection of subgroups. In the case of $V^\Theta(k)$, an answer is as follows.

Lemma 3.4. *Let Θ be a h.s.o.p. of degree $kh + 1$ carrying V^* and let $p \in V^\Theta(k)$ be a point in the parking locus. Let $X(p) \in \mathcal{L}$ be the minimal flat containing p under inclusion and let $d \geq 1$ be minimal such that there exists $w \in W$ with $(w, g^d).p = p$. Then $d|kh$ and the stabilizer $\text{Stab}_{W \times \mathbb{Z}_{kh}}(p)$ is generated by $W_{X(p)}$ and (w, g^d) :*

$$(3.5) \quad \text{Stab}_{W \times \mathbb{Z}_{kh}}(p) = \langle W_{X(p)} \times \{e\}, (w, g^d) \rangle.$$

Proof. In fact, this statement is true for an arbitrary point $p \in V$ and does not depend on p lying in a parking locus.

Since $(w, g^d).p = p$, we have that $(w^m, g^{dm}).p = p$ for all integers $m \geq 1$. The minimality of d forces $d|kh$. Since $p \in X(p)$, any element of the isotropy group $W_{X(p)}$ of the flat $X(p)$ must fix p . This establishes the inclusion \supseteq .

To illustrate the reverse inclusion, suppose $(w', g^{d'}).p = p$ for some $w' \in W$ and some $d' \geq 1$. By our choice of d , there exists an integer m such that $g^{d'} = g^{md}$. Then $p = (w', g^{d'}).p = (w', g^{d'}).p = (w'w^{-m}, e).p$. The group element $w'w^{-m} \in W$ therefore fixes the point $p \in V$. Moreover, we know that the intersection of $X(p)$ with the fixed space $V^{w'w^{-m}}$ is a flat in the intersection lattice \mathcal{L} which contains p . By the minimality of $X(p)$ under inclusion, this forces $X(p) \subseteq V^{w'w^{-m}}$, so that $w'w^{-m} \in W_{X(p)}$. This means that $(w', g^{d'}) = (w'w^{-m}, e)(w, g^d)^m \in \langle W_{X(p)} \times \{e\}, (w, g^d) \rangle$, which proves the inclusion \subseteq . \square

In order to apply Lemma 3.1, the $W \times \mathbb{Z}_{kh}$ -stabilizers of elements in $\text{Park}_W^{NC}(k)$ need to have the same form as the subgroups in Lemma 3.4. To demonstrate this, we have the following result.

Lemma 3.5. *Let $[v, X_1 \supseteq \cdots \supseteq X_k]$ be a k - W -noncrossing parking function and let $d \geq 1$ be minimal such that there exists $w \in W$ with $(w, g^d).[v, X_1 \supseteq \cdots \supseteq X_k] = [v, X_1 \supseteq \cdots \supseteq X_k]$. Then $d|kh$ and the stabilizer $\text{Stab}_{W \times \mathbb{Z}_{kh}}([v, X_1 \supseteq \cdots \supseteq X_k])$ is generated by $W_{vX_1} = vW_{X_1}v^{-1}$ and (w, g^d) :*

$$(3.6) \quad \text{Stab}_{W \times \mathbb{Z}_{kh}}([v, X_1 \supseteq \cdots \supseteq X_k]) = \langle W_{vX_1} \times \{e\}, (w, g^d) \rangle.$$

Proof. We have that $d|kh$ as in the proof of Lemma 3.4. To prove the inclusion \supseteq we need only observe that, for $u \in W_{vX_1}$, we have $v^{-1}uv \in W_{X_1}$. Using the definition of the equivalence relation defining noncrossing parking functions, we compute

$$\begin{aligned} (u, e).[v, X_1 \supseteq \cdots \supseteq X_k] &= [uv, X_1 \supseteq \cdots \supseteq X_k] \\ &= [v(v^{-1}uv), X_1 \supseteq \cdots \supseteq X_k] \\ &= [v, X_1 \supseteq \cdots \supseteq X_k]. \end{aligned}$$

To prove the reverse inclusion, suppose $(w', g^{d'})$ is in the stabilizer on the left hand side. Choose $m \geq 1$ so that $g^{d'} = g^{md}$. Then $(w'w^{-m}, e)$ fixes $[v, X_1 \supseteq \cdots \supseteq X_k]$. In other words, we have $[w'w^{-m}v, X_1 \supseteq \cdots \supseteq X_k] = [v, X_1 \supseteq \cdots \supseteq X_k]$. This implies $w'w^{-m}vW_{X_1} = vW_{X_1}$. Multiplying on the right by v^{-1} gives $w'w^{-m}W_{vX_1} = W_{vX_1}$, or $w'w^{-m} \in W_{vX_1}$. We conclude that $(w', g^{d'}) = (w'w^{-m}, e)(w, g^d)^m \in \langle W_{vX_1} \times \{e\}, (w, g^d) \rangle$. This proves the inclusion \subseteq . \square

By Lemmas 3.4 and 3.5, if p is an element of $\text{Park}_W^{NC}(k)$ or $V^\Theta(k)$, the stabilizer $\text{Stab}_{W \times \mathbb{Z}_{kh}}(p)$ of p inside $W \times \mathbb{Z}_{kh}$ has the form $H = \langle W_X \times \{e\}, (w, g^d) \rangle$, where $X \in \mathcal{L}$ is a fixed flat and $(w, g^d) \in W \times \mathbb{Z}_{kh}$ is a fixed group element. This is a sufficiently well behaved collection of subgroups that we can compute explicit formulas for the corresponding fixed sets in type A and prove that these formulas agree.

In the classical case $k = 1$, Lemma 3.5 can be strengthened somewhat. Let us identify $\mathbb{Z}_h = C = \langle c \rangle$, the subgroup of W generated by our distinguished Coxeter element. The action of $W \times C$ on $\text{Park}_W^{NC}(1)$ is given by $(w, c^m).[v, X] = [wvc^{-m}, c^mX]$. Lemma 3.5 and a quick calculation show that $\text{Stab}_{W \times C}([v, X]) = \langle W_{vX} \times \{e\}, (vc^dv^{-1}, c^d) \rangle$, where $1 \leq d \leq h$ is minimal such that $c^d.X = X$. It would be interesting to see such a structure reflected in the action of $W \times C$ on $V^\Theta(1)$.

4. PARKING ON THE SYMMETRIC GROUP

The goal of this section is to prove the Intermediate Conjecture when $W = \mathfrak{S}_n$ is the symmetric group. Throughout this section we let $W = \mathfrak{S}_n$, we let V be the reflection representation of \mathfrak{S}_n , and we fix a h.s.o.p. $\Theta = (\theta_1, \dots, \theta_{n-1})$ of degree $kh + 1 = kn + 1$ carrying V^* such that the parking locus $V^\Theta(k)$ is reduced.

Our aim is to prove that there exists a $W \times \mathbb{Z}_{kn}$ -equivariant bijection $V^\Theta(k) \cong_{W \times \mathbb{Z}_{kn}} \text{Park}_{\mathfrak{S}_n}^{NC}(k)$. The tool we will use to achieve this is Lemma 3.1. That is, we want to show that for any subgroup $H \leq W \times \mathbb{Z}_{kn}$ which arises as the stabilizer of an element of $V^\Theta(k)$ or $\text{Park}_{\mathfrak{S}_n}^{NC}(k)$, the fixed point sets $V^\Theta(k)^H$ and $\text{Park}_{\mathfrak{S}_n}^{NC}(k)^H$ have the same cardinality. In fact, we will show that these fixed sets are counted by the same formula.

By Lemmas 3.4 and 3.5, we may assume that our subgroup H is given by

$$(4.1) \quad H = \langle W_X \times \{e\}, (w, g^d) \rangle,$$

where X is a flat in the intersection lattice, $w \in \mathfrak{S}_n$ is a permutation, and the positive integer d satisfies $d|kn$. We fix X, w , and d (and hence H) throughout this section. We also fix the notation

$r := \frac{kn}{d}$. We will often identify X with the set partition of $[n]$ defined by $i \sim j$ if and only if the coordinate equality $x_i = x_j$ holds on X .

Thanks to Lemma 3.3, counting the locus fixed set $V^\Theta(k)^H$ is not difficult.

Lemma 4.1. *The fixed set $V^\Theta(k)^H$ has cardinality*

$$(4.2) \quad |V^\Theta(k)^H| = (kn + 1)^{\dim(X \cap E(w, \zeta^{-d}))}.$$

Proof. Let $p \in V^\Theta(k)$. The point p is fixed by the parabolic subgroup W_X if and only if $p \in X$. Also, we have that $(w, g^d).p = \zeta^d(w.p)$, so that (w, g^d) fixes p if and only if $p \in E(w, \zeta^{-d})$. Therefore, we have that

$$(4.3) \quad V^\Theta(k)^H = V^\Theta(k) \cap (X \cap E(w, \zeta^{-d})).$$

By Lemma 3.3, the intersection of subspaces $X \cap E(w, \zeta^{-d})$ is stable under the action of Θ on V . The claimed formula for $|V^\Theta(k)^H|$ follows from Lemma 3.2. \square

The task of the remainder of this section is to prove that we have the corresponding equality $\text{Park}_{\mathfrak{S}_n}^{NC}(k)^H = (kn + 1)^{\dim(X \cap E(w, \zeta^{-d}))}$ for noncrossing parking functions. The strategy is to show that the fixed points $\text{Park}_{\mathfrak{S}_n}^{NC}(k)^H$ are equinumerous with a class of functions having the right cardinality. The argument is reminiscent of various “twelfefold way”-style arguments in enumeration and is a more refined version of arguments appearing in the proof of the Weak Conjecture in type A [2, 26].

Definition 4.2. *Identify the flat X with its corresponding set partition of $[n]$. A function $f : [n] \rightarrow [kn] \cup \{0\}$ is (w, g^d, X) -admissible if*

- $i \sim j$ in X implies $f(i) = f(j)$ and
- $f(w(i)) = g^d(f(i))$ for $1 \leq i \leq n$, where g acts on the set $[kn] \cup \{0\}$ by the permutation $(1, 2, \dots, kn)(0)$.

Given any function $f : [n] \rightarrow [kn] \cup \{0\}$, let $\sigma(f)$ be the set partition of $[n]$ whose blocks are the fibers of f . The first bullet point in Definition 4.2 is the condition that X refines $\sigma(f)$. In particular, the first bullet point is vacuous if $X = V$, in which case Definition 4.2 reduces to the definition of (w, g^d) -admissible functions in [26]. Moreover, if $X \supseteq Y$ are flats, then every (w, g^d, Y) -admissible function is automatically (w, g^d, X) -admissible and a (w, g^d, X) -admissible function f is (w, g^d, Y) -admissible if and only if Y refines $\sigma(f)$.

When $r = \frac{kn}{d} > 1$, the collection of (w, g^d, X) -admissible functions is counted by the same formula as in Lemma 4.1.

Lemma 4.3. *Assume $r > 1$. The number of (w, g^d, X) -admissible functions $f : [n] \rightarrow [kn] \cup \{0\}$ equals the quantity $(kn + 1)^{\dim(X \cap E(w, \zeta^{-d}))}$.*

Proof. The idea is to show that the dimension $\dim(X \cap E(w, \zeta^{-d}))$ may be interpreted in terms of the set partition X and the cycle structure of w . We let e_i denote the i^{th} standard coordinate vector in \mathbb{C}^n for $1 \leq i \leq n$. We have the orthogonal decomposition $\mathbb{C}^n = V \oplus \langle (1, 1, \dots, 1) \rangle$, where V is the reflection representation of \mathfrak{S}_n .

We begin by describing the eigenspace $E(w, \zeta^{-d})$. Let (i_1, i_2, \dots, i_m) be a cycle of the permutation $w \in \mathfrak{S}_n$. The restriction of the operator w on \mathbb{C}^n to the subspace $\langle e_{i_1}, e_{i_2}, \dots, e_{i_m} \rangle$ has the m simple eigenvalues $1, \beta, \beta^2, \dots, \beta^{m-1}$, where $\beta = e^{\frac{2\pi i}{m}}$. In particular, we have that the intersection $(\langle e_{i_1}, e_{i_2}, \dots, e_{i_m} \rangle \cap V) \cap E(w, \zeta^{-d})$ equals 0 unless $r|m$. If $r|m$, the vector $e_{i_1} + \zeta^{-d}e_{i_2} + \dots + \zeta^{-(m-1)d}e_{i_m}$ lies in V and spans the intersection $(\langle e_{i_1}, e_{i_2}, \dots, e_{i_m} \rangle \cap V) \cap E(w, \zeta^{-d})$ (here we are using the assumption $r > 1$). The eigenspace $E(w, \zeta^{-d})$ therefore has one dimension for each cycle of w of length divisible by r , and the eigenvector corresponding to such a cycle (i_1, i_2, \dots, i_m) is $e_{i_1} + \zeta^{-d}e_{i_2} + \dots + \zeta^{-(m-1)d}e_{i_m}$.

Next, let us describe the intersection $X \cap E(w, \zeta^{-d})$. Recall that we have $i \sim j$ in X if and only if the coordinate equality $x_i = x_j$ holds on X . Let (i_1, i_2, \dots, i_m) be a cycle of the permutation w . As before, the subspace $\langle e_{i_1}, e_{i_2}, \dots, e_{i_m} \rangle$ of V corresponding to the cycle (i_1, i_2, \dots, i_m) intersects $X \cap E(w, \zeta^{-d})$ in 0 unless $r|m$. However, if $r|m$, the eigenvector $e_{i_1} + \zeta^{-d}e_{i_2} + \dots + \zeta^{-(m-1)d}e_{i_m} \in E(w, \zeta^{-d})$ of the last paragraph lies in X if and only if for all $1 \leq j, \ell \leq m$ we have the following congruence condition $(*)$ on the entries of the cycle (i_1, i_2, \dots, i_m) :

$$(*) \quad i_j \sim i_\ell \text{ in } X \Rightarrow j - \ell \equiv 0 \pmod{r}.$$

If the congruence condition $(*)$ does not hold on the cycle (i_1, i_2, \dots, i_m) , we still have that $\langle e_{i_1}, e_{i_2}, \dots, e_{i_m} \rangle \cap (X \cap E(w, \zeta^{-d})) = 0$. If $(*)$ does hold on the cycle (i_1, i_2, \dots, i_m) , the intersection $\langle e_{i_1}, e_{i_2}, \dots, e_{i_m} \rangle \cap (X \cap E(w, \zeta^{-d}))$ is one-dimensional. Moreover, if (i_1, i_2, \dots, i_m) and $(i'_1, i'_2, \dots, i'_{m'})$ are two cycles of w such that an element of $\{i_1, i_2, \dots, i_m\}$ is equivalent in X to an element of $\{i'_1, i'_2, \dots, i'_{m'}\}$, the intersection of the larger span $\langle e_{i_1}, e_{i_2}, \dots, e_{i_m}, e_{i'_1}, e_{i'_2}, \dots, e_{i'_{m'}} \rangle$ with $X \cap E(w, \zeta^{-d})$ equals the minimum of the dimensions of $\langle e_{i_1}, e_{i_2}, \dots, e_{i_m} \rangle \cap (X \cap E(w, \zeta^{-d}))$ and $\langle e_{i'_1}, e_{i'_2}, \dots, e_{i'_{m'}} \rangle \cap (X \cap E(w, \zeta^{-d}))$.

We are ready to state the dimension of $X \cap E(w, \zeta^{-d})$ in terms of X and w . Call a cycle (i_1, i_2, \dots, i_m) of w *good* if $r|m$ and the congruence condition $(*)$ holds on (i_1, i_2, \dots, i_m) . Call (i_1, i_2, \dots, i_m) *bad* if it is not good. Define an equivalence relation \sim on cycles of w generated by $C \sim C'$ if an element of C is equivalent to an element of C' in the set partition X . Call an equivalence class of cycles *good* if every cycle in that class is good and *bad* otherwise. The above reasoning gives the following claim.

Claim: The dimension of $X \cap E(w, \zeta^{-d})$ equals the number $G(w)$ of good equivalence classes of cycles of w .

By this claim, we need to show that the number of (w, g^d, X) -admissible functions $f : [n] \rightarrow [kn] \cup \{0\}$ equals $(kn + 1)^{G(w)}$. If f is a (w, g^d, X) -admissible function, the properties $i \sim j$ in $X \Rightarrow f(i) = f(j)$ and $f(w(i)) = g^d(f(i))$ imply that the choice of $f(i)$ for $1 \leq i \leq n$ determines f on the equivalence class of the cycle of w containing i . If (i_1, i_2, \dots, i_m) is a bad cycle of w , we are forced to have $f(i_1) = f(i_2) = \dots = f(i_m) = 0$, so that f sends the entire (bad) equivalence class of (i_1, i_2, \dots, i_m) to 0. On the other hand, if i appears in a cycle of w contained in a good equivalence class we may choose $f(i)$ to be any of the $kn + 1$ elements of $[kn] \cup \{0\}$. The choice of $f(i)$ determines f on the entire (good) equivalence class of the cycle containing i . Since there are $G(w)$ good equivalence classes of cycles, we conclude that there are $(kn + 1)^{G(w)}$ (w, g^d, X) -admissible functions. This completes the proof of the lemma. \square

In the proof of the Weak Conjecture in type A presented in [26], the author characterized set partitions $\sigma(f)$ whose blocks are the fibers of a (w, g^d) -admissible function $f : [n] \rightarrow [kn] \cup \{0\}$. This characterization is easily generalized to include a set partition X .

Definition 4.4. A set partition $\sigma = \{B_1, B_2, \dots\}$ of $[n]$ is (w, r, X) -admissible if

- X refines σ ,
- σ is W -stable in the sense that $w(\sigma) = \{w(B_1), w(B_2), \dots\} = \sigma$,
- at most one block B_{i_0} of σ is itself w -stable in the sense that $w(B_{i_0}) = B_{i_0}$, and
- all other blocks of σ belong to r -element w -orbits.

Observe that if $X \supseteq Y$ are two flats in \mathcal{L} , then every (w, r, Y) -admissible partition σ is automatically (w, r, X) -admissible. In particular, if $X = V$, the first bullet point in Definition 4.4 is vacuous and Definition 4.4 reduces to [26, Definition 8.3].

Lemma 4.5. Assume $r > 1$.

Let $f : [n] \rightarrow [kn] \cup \{0\}$ be a (w, g^d, X) -admissible function. The set partition $\sigma(f)$ of $[n]$ is (w, r, X) -admissible.

The number of (w, g^d, X) -admissible functions $f : [n] \rightarrow [kn] \cup \{0\}$ is the quantity

$$(4.4) \quad \sum_{\sigma} kn(kn - r)(kn - 2r) \cdots (kn - (b_{\sigma} - 1)r),$$

where the sum is over all (w, r, X) -admissible partitions σ of $[n]$ and b_{σ} is the number of orbits of blocks in σ of size r .

The proof of Lemma 4.5 is effectively the same as [26, Lemma 8.4]; one just observes how to take X into account.

Proof. Let $f : [n] \rightarrow [kn] \cup \{0\}$ be a (w, g^d, X) -admissible function and consider the set partition $\sigma(f)$. If the fiber $f^{-1}(0) \subseteq [n]$ is nonempty, it is the unique w -stable block B_{i_0} of $\sigma(f)$ (here we use the assumption $r > 1$). If $f^{-1}(0) = \emptyset$, then $\sigma(f)$ does not contain any w -stable blocks. The condition $i \sim j$ in $X \Rightarrow f(i) = f(j)$ means that X refines $\sigma(f)$. The condition $f(w(i)) = g^d(f(i))$ implies that $\sigma(f)$ is w -stable and the blocks of $\sigma(f)$ (other than B_{i_0} , if it exists) break up into r -element w -orbits.

By [26, Lemma 8.4], the lemma is true when $X = V$. In fact, the proof of [26, Lemma 8.4] shows that the number of (w, g^d, V) -admissible functions $f : [n] \rightarrow [kn] \cup \{0\}$ which induce a fixed (w, r, V) -admissible partition σ of $[n]$ is the product $kn(kn - r)(kn - 2r) \cdots (kn - (b_{\sigma} - 1)r)$. The result for general X follows from the fact that a function $f : [n] \rightarrow [kn] \cup \{0\}$ is (w, g^d, X) -admissible if and only if the associated set partition σ of $[n]$ is (w, r, X) -admissible. \square

Our next goal is to relate admissible set partitions to parking functions. We think of k - \mathfrak{S}_n -noncrossing parking functions as pairs (π, f) , where π is a k -divisible noncrossing partition of $[kn]$ and $f : B \mapsto f(B)$ is a labeling of the blocks of π with subsets of $[n]$ such that $|B| = k|f(B)|$ for every block $B \in \pi$ and $[n] = \bigsqcup_{B \in \pi} f(B)$. To any such pair (π, f) , we associate the set partition $\sigma(\pi, f)$ of $[n]$ defined by $i \sim j$ if i and j label the same block of π under the labeling f .

Lemma 4.6. Assume $r > 1$.

Suppose $(\pi, f) \in \text{Park}_{\mathfrak{S}_n}^{NC}(k)$ is fixed by the subgroup $H = \langle W_X \times \{e\}, (w, g^d) \rangle$. The partition $\sigma(\pi, f)$ of $[n]$ determined by (π, f) is (w, r, X) -admissible.

Conversely, if σ is a fixed (w, r, X) -admissible partition of $[n]$, the number of elements in $\text{Park}_{\mathfrak{S}_n}^{NC}(k)$ which are fixed by H and induce the partition σ is

$$(4.5) \quad kn(kn - r)(kn - 2r) \cdots (kn - (b_{\sigma} - 1)r),$$

where b_{σ} is as in Lemma 4.5.

The hard work here was already done in [26, Lemma 8.5].

Proof. By [26, Lemma 8.5], the lemma is true when $X = V$. To deduce the lemma in general, observe that $(\pi, f) \in \text{Park}_{\mathfrak{S}_n}^{NC}(k)$ is fixed by W_X if and only if the set partition $\sigma(\pi, f)$ associated to (π, f) coarsens X . It follows that if (π, f) is fixed by H , then $\sigma(\pi, f)$ is (w, r, X) -admissible. The claimed product formula is immediate from [26, Lemma 8.5]. \square

We are ready to prove the analog of Lemma 4.1 for noncrossing parking functions.

Lemma 4.7. Let $W = \mathfrak{S}_n$ be of type A . Let $X \in \mathcal{L}$ be a flat in the type A intersection lattice and let $w \in \mathfrak{S}_n$ be a permutation. Find a divisor $d|kn$. The number of k - \mathfrak{S}_n -noncrossing parking functions fixed by the subgroup $H = \langle W_X \times \{e\}, (w, g^d) \rangle$ of $W \times \mathbb{Z}_{kn}$ is given by the formula

$$(4.6) \quad |\text{Park}_{\mathfrak{S}_n}^{NC}(k)^H| = (kn + 1)^{\dim(X \cap E(w, \zeta^d))}.$$

Proof. Assume first that $r > 1$. Lemma 4.3 shows that the right hand side counts the number of (w, g^d, X) -admissible functions $f : [n] \rightarrow [kn] \cup \{0\}$. Lemmas 4.5 and 4.6 show that the left hand side counts the same quantity.

Now consider the case $r = 1$. We have that $\zeta^{-d} = 1$, so that $X \cap E(w, \zeta^d) = X \cap V^w$.

To finish the proof, it suffices to show that $|\text{Park}_{\mathfrak{S}_n}^{NC}(k)^H| = (kn + 1)^{\dim(X \cap V^w)}$. Since $g^d = e$, we have $H = \langle W_X, w \rangle \leq W$. The parking function $(\pi, f) \in \text{Park}_W^{NC}(k)$ is fixed by w if and only if every block B of π is labeled by a union of cycles of w . It follows that (π, f) is fixed by w if and only if (π, f) is fixed by the entire parabolic subgroup W_{V^w} . Therefore, we have that $\text{Park}_W^{NC}(k)^H = \text{Park}_W^{NC}(k)^{H'}$, where $H' = \langle W_X, W_{V^w} \rangle = W_{X \cap V^w} \leq W$.

Consider $X \cap V^w$ as a set partition of $[n]$ and let $w' \in \mathfrak{S}_n$ be a permutation whose cycles are the blocks of $X \cap V^w$. Then $X \cap V^w = V^{w'}$ and $\text{Park}_W^{NC}(k)^{H'} = \text{Park}_W^{NC}(k)^{w'}$. By the \mathfrak{S}_n -equivariant bijection $\text{Park}_{\mathfrak{S}_n}^{NC}(k)$ to classical Fuss parking functions of size n presented in [26], we have $|\text{Park}_W^{NC}(k)^{w'}| = (kn + 1)^{\text{cyc}(w') - 1}$, where $\text{cyc}(w')$ is the number of disjoint cycles of w' . On the other hand, we have $\text{cyc}(w') - 1 = \dim(V^{w'}) = \dim(X \cap V^w)$. \square

All of the pieces are assembled for the proof of the Intermediate Conjecture in type A.

Theorem 4.8. *Let $W = \mathfrak{S}_n$ be of type A and let V denote the associated reflection representation. Let Θ be any h.s.o.p. of degree $kh + 1 = kn + 1$ carrying V^* such that the parking locus $V^\Theta(k)$ is reduced. There exists a $W \times \mathbb{Z}_{kn}$ -equivariant bijection $V^\Theta(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k)$.*

In particular, the Intermediate Conjecture holds in type A and Theorem 1.1 is true.

Proof. Let $H \leq W \times \mathbb{Z}_{kn}$ be a subgroup such that H arises as the $W \times \mathbb{Z}_{kn}$ -stabilizer of some element of either $V^\Theta(k)$ or of $\text{Park}_W^{NC}(k)$. By Lemmas 3.4 and 3.5, there exists a flat $X \in \mathcal{L}$, a group element $w \in W$, and a divisor $d|kn$ such that $H = \langle W_X \times \{e\}, (w, g^d) \rangle$. Lemmas 4.1 and 4.7 may therefore be applied to show that the fixed sets $V^\Theta(k)^H$ and $\text{Park}_W^{NC}(k)^H$ have the same size. Lemma 3.1, with $G = W \times \mathbb{Z}_{kn}$, supplies the desired $W \times \mathbb{Z}_{kn}$ -equivariant bijection. \square

Since we assumed nothing about the h.s.o.p. Θ other than the reducedness of the associated parking locus $V^\Theta(k)$, the argument presented in the last two sections proves the following statement in type A.

Let Θ be any h.s.o.p. of degree $kh + 1$ carrying V^* such that $V^\Theta(k)$ is reduced.

We have a $W \times \mathbb{Z}_{kh}$ -equivariant bijection $V^\Theta(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k)$.

Aside from the claim about the nonempty Zariski open subset \mathcal{U} , this proves the Generalized Strong Conjecture in type A. The same style of argument could be used to prove the above statement for the other classical types BCD, relying on known combinatorial models for $\text{Park}_W^{NC}(k)$ in each of these types.

5. THE GENERIC STRONG CONJECTURE

The purpose of this section is to give evidence for the Generic Strong Conjecture, and in particular show that the Generic Strong and Intermediate Conjectures are equivalent.

Recall that \mathcal{R} is the subset of the affine space $\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ consisting of those W -equivariant polynomial maps $\Theta : V \rightarrow V$ of homogeneous degree $kh + 1$ such that $\Theta^{-1}(0) = \{0\}$ and V^Θ is reduced. Our first main goal is the following result.

Theorem 5.1. *There exists a nonempty Zariski open subset \mathcal{U} of $\text{Hom}_W(V^*, \mathbb{C}[V]_{kh+1})$ such that $\mathcal{U} \subseteq \mathcal{R}$.*

The proof of Theorem 5.1 will rely on Etingof's Theorem 2.3 and will be uniform. By Theorem 5.1, the Main Conjecture exhibits the chain of implications

$$\text{Strong} \Rightarrow \text{Generic Strong} \Rightarrow \text{Intermediate} \Rightarrow \text{Weak},$$

as claimed in the introduction. After proving Theorem 5.1, we will demonstrate that

$$\text{Generic Strong} \Leftrightarrow \text{Intermediate};$$

the proof of this equivalence relies on Theorem 5.1.

To prove Theorem 5.1, we will use a number of basic analytical and topological results. The first of these is a multidimensional version of Rouché's Theorem from complex analysis. Endow \mathbb{C}^N with its usual Euclidean norm $\|\cdot\| : \mathbb{C}^N \rightarrow \mathbb{R}_{\geq 0}$ given by $\|(a_1, \dots, a_N)\| = \sqrt{a_1\bar{a}_1 + \dots + a_N\bar{a}_N}$. For any $z \in \mathbb{C}^N$ and $\epsilon > 0$, let $B(z, \epsilon) = \{z' \in \mathbb{C}^N : \|z - z'\| < \epsilon\}$ be the open Euclidean ball of radius ϵ centered at z .

Let $F : \mathbb{C}^N \rightarrow \mathbb{C}^N$ be a holomorphic function. A *zero* of F is a point $z_0 \in \mathbb{C}^N$ with $F(z_0) = 0$. A zero z_0 of F is called *isolated* if there exists $\epsilon > 0$ such that z_0 is the only zero of F contained in the ball $B(z_0, \epsilon)$. Given an isolated zero z_0 of F , one can define the multiplicity m of z_0 using the Taylor expansion of F around z_0 . An isolated zero z_0 of F is called *simple* if it has multiplicity one; this is equivalent to the condition that the Jacobian matrix of F is nondegenerate at z_0 .

In this paper, we will only consider the case where F is a polynomial mapping and all of its zeros z_0 are simple. As in the case $N = 1$, the multidimensional version of Rouché's Theorem gives control over the number of zeros contained in some bounded domain Ω under perturbations of F which are “small” on the boundary $\partial\Omega$.

Theorem 5.2. (*Multidimensional Rouché Theorem*) *Let $F, G : \mathbb{C}^N \rightarrow \mathbb{C}^N$ be holomorphic functions and let $\Omega \subset \mathbb{C}^N$ be a bounded domain whose boundary $\partial\Omega$ is homeomorphic to a sphere. Assume that F has a finite number of zeros on the domain Ω (which will automatically be isolated). Let M be the number of these zeros, counted with multiplicity. If the inequality $\|F\| > \|G\|$ holds on the boundary $\partial\Omega$, then the function $F + G$ also has M zeros on Ω , counted with multiplicity.*

In all of our applications of Theorem 5.2, the functions F and G will be polynomial, the domain Ω will be a ball, and we will have the zero count $M = 1$.

To prove Theorem 5.1 and the equivalence of the Generalized Strong and Intermediate Conjectures, we will need to consider the relationship between the Euclidean and Zariski topologies on the set \mathbb{C}^N . The first tool we use in this regard is well known.

Lemma 5.3. *Let $\mathcal{U} \subseteq \mathbb{C}^N$ be a nonempty Zariski open set and suppose $\mathcal{X} \subset \mathbb{C}^N$ satisfies $\mathcal{U} \subseteq \mathcal{X} \subseteq \mathbb{C}^N$. Then \mathcal{X} is path connected.*

Proof. As a Zariski open subset of \mathbb{C}^N , we know that \mathcal{U} is path connected. Let $\mathcal{V} := \mathbb{C}^N - \mathcal{U}$ be the complement of \mathcal{U} in \mathbb{C}^N , so that \mathcal{V} is a proper subvariety of \mathbb{C}^N . It is enough to show that for any point $p \in \mathcal{V}$, there exists a path $\gamma : [0, 1] \rightarrow \mathbb{C}^N$ such that $\gamma(0) = p$ and $\gamma(t) \in \mathcal{U}$ for $0 < t \leq 1$.

We claim that we can take γ to be a linear path $\gamma(t) = p + v_0 t$ for some $v_0 \in \mathbb{C}^N - \{0\}$. To see this, let $L_v := \{p + vz : z \in \mathbb{C}\}$ for $v \in \mathbb{C}^N - \{0\}$. Then L_v is a copy of \mathbb{C} and $\mathcal{V} \cap L_v$ is a subvariety of L_v . It follows that $\mathcal{V} \cap L_v$ is finite or $\mathcal{V} \subseteq L_v$ for any $v \in \mathbb{C}^N - \{0\}$. Since \mathcal{V} is a proper subvariety of \mathbb{C}^N , we can find $v' \in \mathbb{C}^N - \{0\}$ such that $\mathcal{V} \cap L_{v'}$ is finite. This means that there exists a nonzero complex number α such that $p + \alpha v' t \in \mathcal{U}$ for all real numbers $0 < t \leq 1$. Taking $v_0 := \alpha v'$, we get our desired path γ . \square

Recall that a subset $\mathcal{C} \subseteq \mathbb{C}^N$ is called *constructible* if there exist varieties $\mathcal{V}_1, \dots, \mathcal{V}_m, \mathcal{W}_1, \dots, \mathcal{W}_n \subseteq \mathbb{C}^N$ such that

$$(5.1) \quad \mathcal{C} = \bigcup_{i=1}^m (\mathcal{V}_i - \mathcal{W}_i).$$

Equivalently, a subset $\mathcal{C} \subseteq \mathbb{C}^N$ is constructible if it is locally closed in the Zariski topology. We will need to consider images of varieties under the standard projection map $\pi : \mathbb{C}^{N+n} \rightarrow \mathbb{C}^N$. While such images are not varieties in general, we have the following result (see, for example, [9]).

Lemma 5.4. *Let $\pi : \mathbb{C}^{N+n} \rightarrow \mathbb{C}^N$ be the projection map obtained by forgetting the last n coordinates. If $\mathcal{V} \subseteq \mathbb{C}^{N+n}$ is a variety, then $\pi(\mathcal{V}) \subseteq \mathbb{C}^N$ is a constructible set.*

It will be crucial for us to show that a certain constructible set \mathcal{C} has nonempty Zariski interior. To do this, we will use the following fact.

Lemma 5.5. *Let $\mathcal{C} \subseteq \mathbb{C}^N$ be a constructible set and let \mathcal{U} be the Zariski interior of \mathcal{C} . Suppose there is a point $p \in \mathcal{C}$ and a real number $\delta > 0$ such that the open ball $B_{\mathbb{C}^N}(p, \delta)$ satisfies $B_{\mathbb{C}^N}(p, \delta) \subseteq \mathcal{C}$. Then \mathcal{U} is nonempty and contains p .*

Proof. Since \mathcal{C} is constructible, if $\mathcal{U} = \emptyset$ then \mathcal{C} would be contained in a proper subvariety of \mathbb{C}^N . But no proper subvariety of \mathbb{C}^N contains a nonempty Euclidean ball, so \mathcal{U} is nonempty and $p \in \mathcal{U}$. \square

We have assembled all the pieces we need to prove Theorem 5.1.

Proof. (of Theorem 5.1) We start by enlarging our ambient space to consider all homogeneous degree $kh + 1$ polynomial maps $\Theta : V \rightarrow V$, whether or not they are W -equivariant.

For the remainder of this proof, fix a choice of ordered basis x_1, \dots, x_n of the dual space V^* of the reflection representation. Let \mathbb{A} denote the product

$$(5.2) \quad \mathbb{A} = \overbrace{\mathbb{C}[x_1, \dots, x_n]_{kh+1} \times \cdots \times \mathbb{C}[x_1, \dots, x_n]_{kh+1}}^n = \overbrace{\mathbb{C}[V]_{kh+1} \times \cdots \times \mathbb{C}[V]_{kh+1}}^n.$$

Counting monomials, we get that \mathbb{A} is a copy of the affine complex space \mathbb{C}^N , where $N = \binom{kh+n}{kh+1}^n$. For every point $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{A}$, we get an associated polynomial mapping $\Theta : V \rightarrow V$ which sends a point with coordinates (x_1, \dots, x_n) to the point with coordinates $(\theta_1, \dots, \theta_n)$. This identifies \mathbb{A} with the collection of homogeneous polynomial maps $V \rightarrow V$ of degree $kh + 1$.

We claim that there is a nonempty Zariski open subset \mathcal{V} of \mathbb{A} such that, for every mapping $\Theta : V \rightarrow V$ in \mathcal{V} , we have $\Theta^{-1}(0) = \{0\}$. To see this, let $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{A}$. It is well known that $\Theta^{-1}(0) = \{0\}$ if the sequence $\theta_1, \dots, \theta_n \in \mathbb{C}[V]_{kh+1}$ is a regular sequence in the polynomial ring $\mathbb{C}[V]$. So it is enough to show that there exists a nonempty Zariski open subset $\mathcal{V} \subset \mathbb{A}$ such that for all $\Theta = (\theta_1, \dots, \theta_n) \in \mathcal{V}$, the sequence $\theta_1, \dots, \theta_n$ is regular. This is a well known fact in algebra; see for example [27, p. 48].

Given any $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{A}$, let $V^\Theta(k)$ be the subscheme of V cut out by the ideal $(\Theta - \mathbf{x}) := (\theta_1 - x_1, \dots, \theta_n - x_n)$. This is the same definition as before, but we are no longer assuming that the linear map $x_i \mapsto \theta_i$ is W -equivariant.

Given $\Theta \in \mathcal{V}$, the reducedness of $V^\Theta(k)$ can be detected by a Jacobian condition. Let $\text{Mat}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices. Given any point $v \in V$ with coordinates $(x_1(v), \dots, x_n(v)) = (v_1, \dots, v_n) \in \mathbb{C}^n$, let $J(\Theta)_v \in \text{Mat}_n(\mathbb{C})$ denote the Jacobian matrix $\left(\frac{\partial \theta_i}{\partial x_j} \right)_{1 \leq i, j \leq n}$ of the polynomial map $\Theta = (\theta_1, \dots, \theta_n) : V \rightarrow V$ evaluated at the point $(x_1, \dots, x_n) = (v_1, \dots, v_n)$. The Jacobian criterion for reducedness is as follows.

For $\Theta \in \mathcal{V}$, the scheme $V^\Theta(k)$ fails to be reduced if and only if there is some point $v \in V$ such that $\Theta(v) = v$ and the matrix $J(\Theta)_v$ has 1 as an eigenvalue.

Equivalently, if $I_n \in \text{Mat}_n(\mathbb{C})$ denotes the $n \times n$ identity matrix, then $V^\Theta(k)$ fails to be reduced if any only if there is some point $v \in V$ such that $\Theta(v) - v = 0$ and the matrix $J(\Theta)_v - I_n$ is singular.

The reasoning of the last paragraph leads us to consider the diagram of maps

$$(5.3) \quad \mathbb{A} \xleftarrow{\pi} \mathbb{A} \times V \xrightarrow{\varphi} \text{Mat}_n(\mathbb{C}) \times V,$$

where the map on the left is projection $\pi : \mathbb{A} \times V \rightarrow \mathbb{A}$ onto the first factor and the map on the right is $\varphi : (\Theta, v) \mapsto (J(\Theta)_v - I_n, \Theta(v) - v)$. Both π and φ are morphisms of affine complex varieties. Consider the subvariety $\mathcal{Z} \subset \text{Mat}_n(\mathbb{C}) \times V$ given by

$$(5.4) \quad \mathcal{Z} = \{A \in \text{Mat}_n(\mathbb{C}) : \det(A) = 0\} \times \{0\}.$$

The Jacobian criterion for reducedness translates to say:

Given $\Theta \in \mathcal{V} \subset \mathbb{A}$, the scheme $V^\Theta(k)$ fails to be reduced if and only if $\Theta \in \pi(\varphi^{-1}(\mathcal{Z}))$.

We have that $\varphi^{-1}(\mathcal{Z})$ is a subvariety of the product space $\mathbb{A} \times V$. Since π is projection $\mathbb{C}^{N+n} \rightarrow \mathbb{C}^N$, the “pathological set” $\pi(\varphi^{-1}(\mathcal{Z}))$ is a constructible subset of \mathbb{A} by Lemma 5.4.

At this point in the proof, we turn our attention to W -equivariant maps. In order to do this, define a subset $\mathbb{A}^W \subset \mathbb{A}$ by

$$(5.5) \quad \mathbb{A}^W := \{(\theta_1, \dots, \theta_n) \in \mathbb{A} : \text{the linear map } x_i \mapsto \theta_i \text{ is } W\text{-equivariant}\}.$$

Then \mathbb{A}^W is a linear subvariety of \mathbb{A} . We may identify \mathbb{A}^W with $\text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$, so we may embed $\mathcal{R} \subset \mathbb{A}^W$.

It is our aim to show that the subset $\mathcal{R} \subset \mathbb{A}^W$ has nonempty Zariski interior. We know that \mathcal{R} may be expressed as

$$(5.6) \quad \mathcal{R} = (\mathcal{V} - \pi(\varphi^{-1}(\mathcal{Z}))) \cap \mathbb{A}^W.$$

Since $\pi(\varphi^{-1}(\mathcal{Z}))$ is a constructible subset of \mathbb{A} , we have that \mathcal{R} is a constructible subset of \mathbb{A}^W . By Etingof’s Theorem 2.3, we know that \mathcal{R} is nonempty; choose $\Theta_0 \in \mathcal{R}$. Let $\mathcal{U} \subseteq \mathcal{R}$ be the Zariski interior of \mathcal{R} within \mathbb{A}^W . We use Lemma 5.5 to argue that \mathcal{U} is nonempty as follows.

Equip the affine space \mathbb{A} with its standard Euclidean metric. Then \mathbb{A}^W inherits this metric from \mathbb{A} . By Lemma 5.5, to show that $\mathcal{U} \neq \emptyset$ it suffices to show

there exists $\delta > 0$ such that for all $\Theta = (\theta_1, \dots, \theta_n) \in B_{\mathbb{A}^W}(\Theta_0, \delta)$, the system of equations $\theta_1 - x_1 = \dots = \theta_n - x_n = 0$ has precisely $(kh+1)^n$ solutions in $V = \mathbb{C}^n$, all of them simple.

This is a purely analytical statement and can be seen from Theorem 5.2. Let us temporarily identify V with \mathbb{C}^n . For any $\Theta \in \mathbb{A}^W$, introduce the function $F_\Theta : \mathbb{C}^n \rightarrow \mathbb{C}^n$ whose coordinates are given by $F_\Theta = (\theta_1 - x_1, \dots, \theta_n - x_n)$. With this notation, the holomorphic function F_{Θ_0} has $(kh+1)^n$ simple zeros in \mathbb{C}^n . Let $\epsilon > 0$ denote the minimum distance between any pair of these zeros. Let $K \subset \mathbb{C}^n$ denote the compact set

$$(5.7) \quad K = \bigcup_v \left\{ z \in \mathbb{C}^n : \|z - v\| = \frac{\epsilon}{100} \right\},$$

where the union is over all $(kh+1)^n$ solutions $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ to $F_{\Theta_0}(v) = 0$. By compactness and the fact that F_{Θ_0} is nonvanishing on K , there exists $m > 0$ such that $\|F_{\Theta_0}(k)\| > m$ for all $k \in K$. Again by compactness, there exists $\delta > 0$ such that for all $\Theta \in \mathbb{A}^W$ with $\Theta \in B_{\mathbb{A}^W}(\Theta_0, \delta)$, we have that

$$(5.8) \quad \sup \{ \|F_\Theta(z) - F_{\Theta_0}(z)\| : z \in K \} < \frac{m}{100}.$$

Let $\Theta \in B_{\mathbb{A}^W}(\Theta_0, \delta)$. Let $v \in \mathbb{C}^n$ be a zero of F_{Θ_0} . Then just one zero of F_{Θ_0} lies in the ball $\|z - v\| \leq \frac{\epsilon}{100}$, and that zero is simple. Since $\|F_\Theta - F_{\Theta_0}\| < \frac{m}{100} < \|F_{\Theta_0}\|$ on the boundary of this ball, the Theorem 5.2 tells us that F_Θ has the same number of zeros (counted with multiplicity) in this ball as F_{Θ_0} . We conclude that F_Θ has exactly one zero in this ball, and that zero is simple. Since our choice of zero v was arbitrary, we get that F_Θ has at least $(kh+1)^n$ simple zeros in \mathbb{C}^n . By Bézout’s Theorem (or another application of Theorem 5.2), we know that F_Θ has precisely $(kh+1)^n$ zeros in \mathbb{C}^n , all of them simple.

The last paragraph shows that the constructible set \mathcal{R} contains a Euclidean open ball centered at Θ_0 , and so must have nonempty Zariski interior \mathcal{U} by Lemma 5.5. This completes the proof of Theorem 5.1. \square

Our final task is to prove Theorem 1.2.

Theorem 1.3. *The Intermediate and Generalized Strong Conjectures are equivalent.*

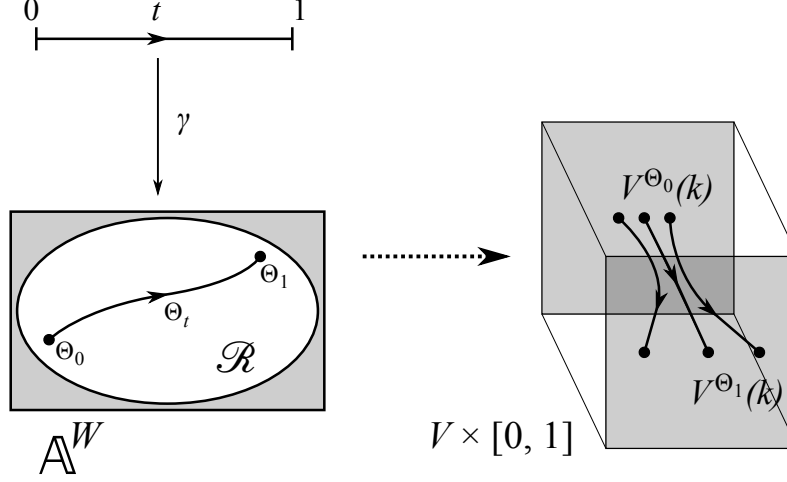


FIGURE 1. The proof of Theorem 1.2.

Proof. (of Theorem 1.2) In light of Etingof's Theorem 2.3, the Generalized Strong Conjecture certainly implies the Intermediate Conjecture. Let us assume that the Intermediate Conjecture is true and derive the Generalized Strong Conjecture.

Fix an h.s.o.p. $\Theta_0 \in \mathcal{R}$ of degree $kh + 1$ carrying V^* such that the parking locus $V^{\Theta_0}(k)$ is reduced and we have a $W \times \mathbb{Z}_{kh}$ -equivariant bijection $V^{\Theta_0}(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k)$. Let $\Theta_1 \in \mathcal{R}$ be another h.s.o.p. of degree $kh + 1$ carrying V^* such that $V^{\Theta_1}(k)$ is reduced. It is enough to show that we have a $W \times \mathbb{Z}_{kh}$ -equivariant bijection $V^{\Theta_1}(k) \cong_{W \times \mathbb{Z}_{kh}} \text{Park}_W^{NC}(k)$.

By Theorem 5.1, the subset $\mathcal{R} \subset \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$ has nonempty Zariski interior \mathcal{U} . By Lemma 5.3, this means that \mathcal{R} is path connected in its Euclidean topology. Let $\gamma : [0, 1] \rightarrow \mathcal{R}$ be a path $\gamma : t \mapsto \Theta_t$ from Θ_0 to Θ_1 in \mathcal{R} . For all real numbers $0 \leq t \leq 1$, we get an associated parking locus $V^{\Theta_t}(k) \subset V$ consisting of $(kh + 1)^n$ distinct points.

Let us consider the action of the group $W \times \mathbb{Z}_{kh}$ on V . For any group element $(w, g^d) \in W \times \mathbb{Z}_{kh}$, the function $V \rightarrow V$ given by $v \mapsto (w, g^d).v$ is continuous. Moreover, for all $0 \leq t \leq 1$, the $(kh + 1)^n$ element set $V^{\Theta_t}(k) \subset V$ is closed under the action of $W \times \mathbb{Z}_{kh}$. The idea is to follow the action of a fixed group element (w, g^d) along the path γ .

We claim that there exist unique (continuous) paths $\alpha_v : [0, 1] \rightarrow V$ for $v \in V^{\Theta_0}(k)$ such that $\alpha_v(0) = v$ and $\alpha_v(t) \in V^{\Theta_t}(k)$ for all $0 \leq t \leq 1$. To see this, consider the locus $V^{\Theta_t}(k)$ for some fixed $0 \leq t \leq 1$. Let ϵ_t be the minimum distance between any pair of points in this $(kh + 1)^n$ -element set. By Theorem 5.2, there exists an open interval I_t in $[0, 1]$ containing t such that for all $t' \in I_t$ and $v^{(t)} \in V^{\Theta_t}(k)$, there is a unique point $v^{(t')} \in V^{\Theta_{t'}}(k) \cap B_V(v^{(t)}, \frac{\epsilon_t}{100})$. For any point $v^{(t)} \in V^{\Theta_t}(k)$, we therefore get a well defined function $\alpha_{v^{(t)}}^{(t)} : I_t \rightarrow V$ given by $t' \mapsto v^{(t')}$. Theorem 5.2 also shows that $\alpha_{v^{(t)}}^{(t)}$ is continuous for all $v^{(t)} \in V^{\Theta_t}(k)$, and our assumption on I_t guarantees that the set of functions $\{\alpha_{v^{(t)}}^{(t)} : I_t \rightarrow V : v^{(t)} \in V^{\Theta_t}(k)\}$ is the unique collection of continuous functions $I_t \rightarrow V$ such that $\alpha_{v^{(t)}}^{(t)}(t') \in V^{\Theta_{t'}}(k)$ and $\alpha_{v^{(t)}}^{(t)}(t) = v^{(t)}$ for all $v^{(t)} \in V^{\Theta_t}(k)$ and $t' \in I_t$. By compactness, there are finitely many $I_{t_1}, I_{t_2}, \dots, I_{t_m}$ of these open intervals which cover $[0, 1]$. There exists a unique collection of functions $\{\alpha_v : [0, 1] \rightarrow V : v \in V^{\Theta_0}(k)\}$ such that $\alpha_v(0) = v$ and $\alpha_v|_{I_{t_r}} = \alpha_{v^{(t_r)}}$ for some $v^{(t_r)} \in V^{\Theta_{t_r}}(k)$ and all $1 \leq r \leq m$. The functions α_v are continuous.

The above reasoning implies that $V^{\Theta_t}(k) = \{\alpha_v(t) : v \in V^{\Theta_0}(k)\}$ for all $0 \leq t \leq 1$. By compactness, there exists $\epsilon > 0$ such that $\|\alpha_v(t) - \alpha_{v'}(t)\| > \epsilon$ for all $v \neq v'$ and all $0 \leq t \leq 1$. Let $(w, g^d) \in W \times \mathbb{Z}_{kh}$ and suppose $(w, g^d).v = v'$ for $v, v' \in V^{\Theta_0}(k)$. The continuity of the

action of (w, g^d) on V and the continuity of the α paths means that $(w, g^d) \cdot \alpha_v(t) = \alpha_{v'}(t)$ for all $0 \leq t \leq 1$. Therefore, the map $\alpha_v(0) \mapsto \alpha_v(1)$ gives the desired $W \times \mathbb{Z}_{kh}$ -equivariant bijection $V^{\Theta_0}(k) \rightarrow V^{\Theta_1}(k)$. \square

The geometric intuition behind the previous proof is shown in Figure 1. Suppose we are given two h.s.o.p.'s Θ_0 and Θ_1 of degree $kh + 1$ carrying V^* such that the loci $V^{\Theta_0}(k)$ and $V^{\Theta_1}(k)$ are both reduced. We think of Θ_0 and Θ_1 as points in the affine space $\mathbb{A}^W = \text{Hom}_{\mathbb{C}[W]}(V^*, \mathbb{C}[V]_{kh+1})$. We identify the parameter space of all h.s.o.p.'s Θ of degree $kh + 1$ such that $V^{\Theta}(k)$ is reduced with $\mathcal{R} \subset \mathbb{A}^W$, so that $\Theta_0, \Theta_1 \in \mathcal{R} \subset \mathbb{A}^W$.

We want to show that the parking loci $V^{\Theta_0}(k)$ and $V^{\Theta_1}(k)$ have the same $W \times \mathbb{Z}_{kh}$ -set structure. To do this, we start by connecting the h.s.o.p.'s Θ_0 and Θ_1 with a path $\gamma : [0, 1] \rightarrow \mathbb{A}^W$ whose image lies entirely within the parameter space \mathcal{R} of reduced h.s.o.p.'s. By Theorem 5.1 and Lemma 5.3, the space \mathcal{R} is path connected so that this can be accomplished.

For every real number $0 \leq t \leq 1$, the path γ gives us a h.s.o.p. $\gamma(t) = \Theta_t$ of degree $kh + 1$ carrying V^* with the property that $V^{\Theta_t}(k)$ is reduced. For any value of t , we therefore get a subset $V^{\Theta_t}(k) \subset V$ which consists of $(kh + 1)^n$ points and is stable under the continuous action of $W \times \mathbb{Z}_{kh}$ on V . The right of Figure 1 shows the loci $V^{\Theta_t}(k)$ inside the product space $V \times [0, 1]$ as $t \in [0, 1]$ varies; in this case we have $(kh + 1)^n = 3$. The reducedness assumption means that the $(kh + 1)^n$ points in $V^{\Theta_t}(k)$ trace out $(kh + 1)^n$ paths in V and *these paths never merge*. The continuity of the action of $W \times \mathbb{Z}_{kh}$ on V means that following our group action along these disjoint paths must give us the desired $W \times \mathbb{Z}_{kh}$ -set isomorphism $V^{\Theta_0}(k) \cong_{W \times \mathbb{Z}_{kh}} V^{\Theta_1}(k)$.

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